THE UNIVERSITY OF ALBERTA

STUDIES ON NONLINEAR SAMPLED-DATA

SYSTEMS WITH NON-ZERO INITIAL CONDITIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ELECTRICAL ENGINEERING

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STUDIES ON NONLINEAR DISCRETE-DATA SYSTEMS WITH NON-ZERO INITIAL CONDITIONS

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© AHMED MOHAMED HELMY RASHED

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Studies on Nonlinear Discrete-Data Systems with Non-Zero Initial Conditions" submitted by Ahmed M.H. Rashed in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

Nonlinear discrete-data systems subject to non-zero initial conditions are considered. Two types of systems are investigated; the first has no hold device, while the second includes a zero order hold. In both systems the nonlinear element is located in the feedback path. Linear discrete-data systems with non-zero initial conditions are considered first and then it is shown how one, similarly, can treat nonlinear discrete-data systems with non-zero initial conditions. In the latter case system response takes the form of the discrete Volterra series in which non-zero initial conditions are included. The series is obtained by solving the system equation by iteration. Uniqueness as well as convergence of the series, and consequently bounded-input bounded-output stability of the nonlinear system, are investigated by use of the contraction mapping principle. By letting the system input be zero, asymptotic stability of the system can be studied. Finally, the intersampling response of nonlinear discrete-data systems is obtained in the form of an infinite series, and the stability of the system between sampling instants is discussed.

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CHAPTER I

INTRODUCTION

A control system in which the signal at one or more points is in the form of pulses occurring at discrete instants of time is called a sampled-data or a discrete-data system. If the time interval between every two consecutive pulses is a constant, the signal is said to be uniformly sampled. A typical sampled-data system utilizing uniform sampling is shown in Fig. 1.1.

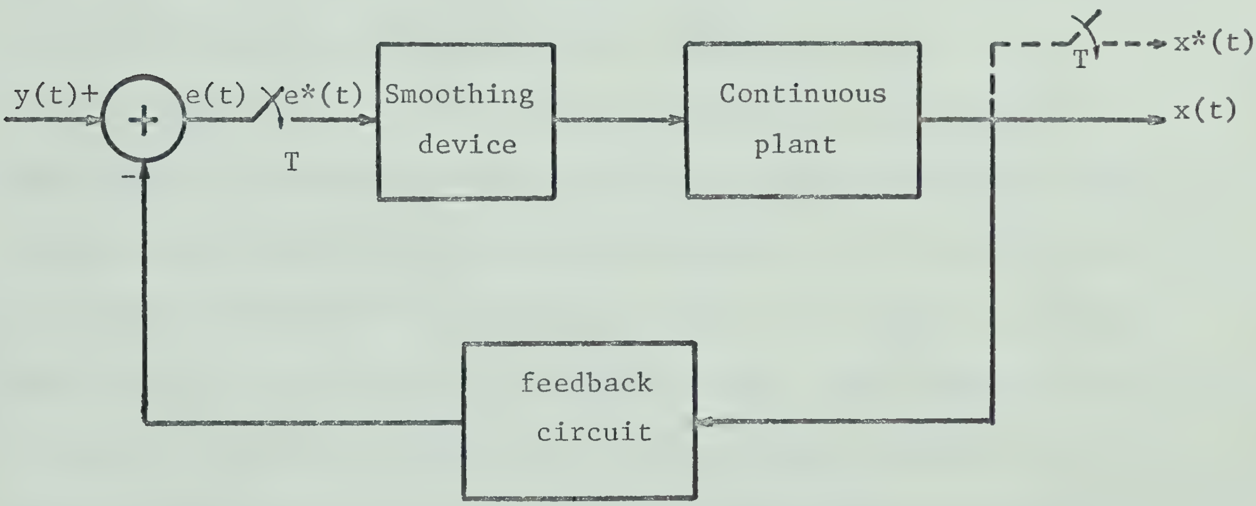

Fig. 1.1 Typical Sampled-data System

Discrete-data systems have drawn the attention of many investigators since the Second World War. The main reason is the rapidly growing use of digital equipment, mainly digital computers, in

modern scientific and technological applications. Examples are: computer-controlled machine tools, time sharing of transmission links, ground-controlled guided missiles, etc.

An important field of investigation into the behaviour of these systems is system stability. However, much of the work done in this area so far discusses only the problem of asymptotic stability[1-12*]. Of the most significant work in this context is that of Tsypkin[5-8], who extended the results of V.M. Popov for nonlinear continuous-data systems to discrete-data systems. Jury and Lee[9-12] further extended the work of Tsypkin, and by constraining the slope of the nonlinearity, they were able to obtain less conservative results.

Another important concept of system stability is the concept of bounded-input bounded-output (BIBO) stability. Here a system is said to be stable in the BIBO sense if, and only if, every bounded input produces a bounded output. For linear systems the equivalence between system stability in the BIBO sense and in the Liapunov sense has been established[13,14]. On the other hand, although it has been shown, for a certain class of nonlinear systems, that conditions sufficient for establishing asymptotic stability also establish BIBO stability[15-18], there is, in general, no simple equivalence between these two concepts and counter examples have been constructed[13].

Volterra series provides a powerful tool for finding system response and system stability of nonlinear systems[19-24]. The convergence of the series implies that the system is stable.

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* Numbers placed above the line of text refer to references.

It is the objective of this thesis to use Volterra series to study nonlinear discrete-data systems with non-zero initial conditions. The response of the system at sampling instants, as well as in between sampling instants, is represented by a discrete Volterra-type series. BIBO stability as well as asymptotic stability are then studied through the convergence of the resulting series utilizing the Banach fixed point theorem[25-28].

A brief review of the important results obtained so far in the area of system stability is given in chapter II. In chapter III linear discrete-data systems are considered. Non-zero initial conditions are included in a general expression for system response. Both BIBO and asymptotic stability are discussed using functional representation of the system. The functional notation and concepts involved are introduced in that chapter. In chapter IV the development of chapter III is extended to nonlinear discrete-data systems. The response of the system is obtained in the form of a discrete Volterra-type series by a process of successive approximations. Chapter V deals with the response of the nonlinear discrete-data systems between sampling instants.

A brief review of Z-transform theory, which is the main tool used to associate initial conditions with system response, is given in the appendix.

CHAPTER II

REVIEW OF DISCRETE-DATA SYSTEM STABILITY

2.1 Introduction

In considering the stability of a given control system we often consider both asymptotic and bounded-input bounded-output (BIBO) stability. An undriven system is said to be asymptotically stable if, given some set of initial conditions, its output tends to zero as time tends to infinity. On the other hand a driven system is said to be BIBO stable if its output remains bounded for each bounded input. However, while it is true that for linear systems asymptotic stability implies BIBO stability, this is not, in general, true for nonlinear systems[13].

The purpose of this chapter is to review briefly the important results obtained in the field of stability of nonlinear discrete-data systems. The first section discusses asymptotic stability results and the second BIBO stability results.

2.2 Asymptotic Stability of Nonlinear Discrete-Data Systems

The first extensive work in the field of asymptotic stability of nonlinear discrete-data systems is the work of Hahn[1,2] and Kalman and Bertram[3*]. They extended the ideas of the second method of Liapunov to apply to systems of difference equations. The results they obtained governing the choice of Liapunov functions suitable for different types of stability of the null solution are analogous to

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* See also Ref. 4.

those obtained for continuous-data systems. Although this method is very general, its application requires both intuition and experience in the choice of suitable functions.

The same problem was treated by the Russian scientist Ya.Z. Tsypkin who has shown that it can be solved more easily and effectively by working in the frequency domain[5-7]. He extended the ideas of the Rumanian scientist V.M. Popov for the determination of the stability in-the-large of continuous autonomous systems to discrete-data systems. This method yields results which are intimately connected with the usual notion of frequency response characteristics and, more importantly, gives general sufficient conditions for stability which are applicable to systems of arbitrary order.

The system considered by Tsypkin consists of a nonlinear element located in the forward path of an otherwise linear feedback system, Fig. 2.1(a). The nonlinear characteristics $\phi(x(n))$ is assumed to be memoryless and confined to a specified gain sector $[0, k)$, Fig. 2.1(b). Mathematically the nonlinearity is assumed to satisfy the conditions:

$$\begin{aligned} &a) \quad \phi(0) = 0 \\ &b) \quad 0 \leq \frac{\phi(x(n))}{x(n)} < k \end{aligned} \tag{2.1}$$

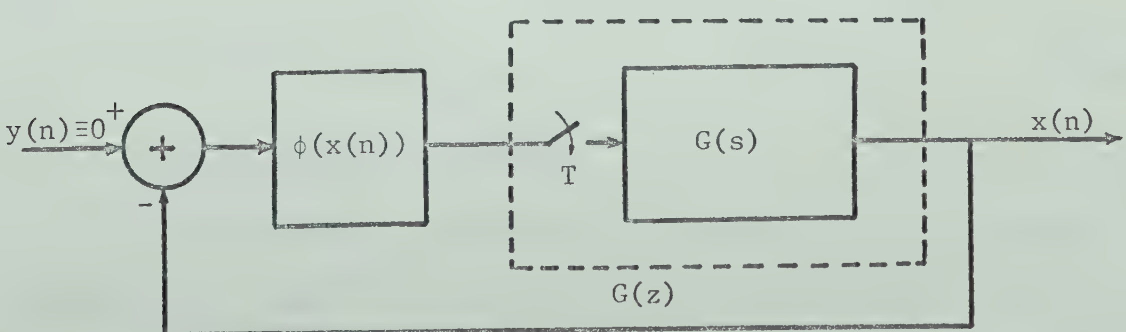

Fig. 2.1 (a)

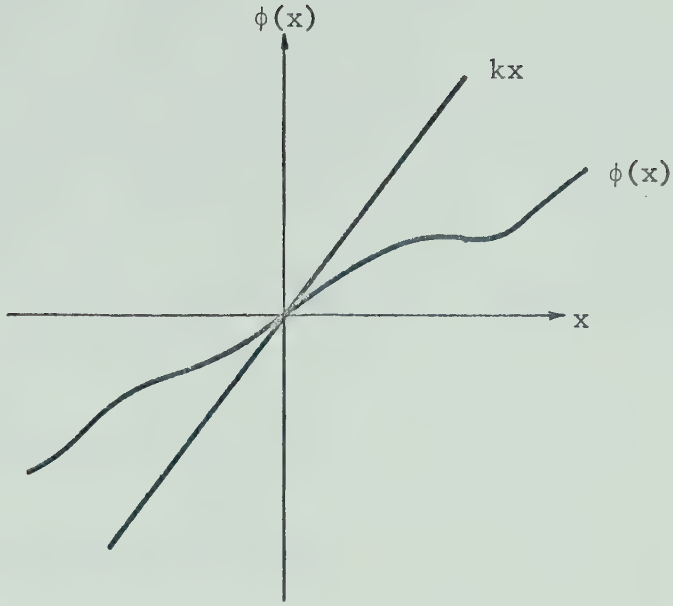

Fig. 2.1 (b)

Fig. 2.1 Nonlinear discrete-data system considered by Tsypkin
(a) system configuration
(b) nonlinearity characteristics

The results obtained by Tsypkin are similar in many respects to the Nyquist criterion for the stability of linear discrete-data systems. However, they give only sufficient conditions for system stability. The system of Fig. 2.1 is asymptotically stable in the large if the following inequality is satisfied:

$$\mathrm{Re}[G(e^{j\omega T})] + \frac{1}{k} > 0 \tag{2.2}$$

where

$$G(e^{j\omega T}) = G(z)\Big|_{z=e^{j\omega T}} \tag{2.3}$$

T is the sampling period.

Geometrically, inequality (2.2) means that the frequency characteristics $G(e^{j\omega T})$ of the linear part must be located to the right of a vertical straight line crossing the abscissa of the $G(e^{j\omega T})$

plane at the point $(-\frac{1}{k}, 0)$.

Tsypkin further extended the class of systems for which inequality (2.2) gives sufficient conditions for stability. Time-varying nonlinearities and systems with neutral or unstable linear part are included. In this case the nonlinearity, instead of satisfying condition (2.1 b), has to satisfy the condition:

$$\varepsilon < \frac{\phi(x(n),n)}{x(n)} < k + \varepsilon \tag{2.4}$$

where ε is a real number such that

$$\hat{G}(z) = \frac{G(z)}{1+\varepsilon G(z)} \tag{2.5}$$

is stable; and $\hat{G}(e^{j\omega T})$ takes the place of $G(e^{j\omega T})$ in (2.2).

Tsypkin also noted that for some cases condition (2.2) is not only sufficient but also necessary for asymptotic stability. These cases arise when the maximum negative real part of $G(e^{j\omega T})$, or $\hat{G}(e^{j\omega T})$, is attained at the abscissa. This is because condition (2.2) in this case coincides with the necessary and sufficient conditions of stability of the linearized system.

Tsypkin, in a later paper[8], formulated his results into the familiar circle criterion.

The work of Tsypkin was further extended by Jury and Lee[9-10]. They considered the same class of systems considered by Tsypkin with the addition that the nonlinearity is to satisfy the extra condition:

$$\left|\frac{d\phi(x(n))}{dx(n)}\right| < k' < \infty \tag{2.6}$$

or more generally

$$k_1 < \frac{d\phi(x(n))}{dx(n)} < k_2 \tag{2.7}$$

where k' is a positive real number; and k_1 and k_2 are any real numbers.

The approach used by Jury and Lee is also a frequency domain approach and, like Tsypkin's approach, depends on the frequency characteristics of the linear plant which is assumed to have all of its poles in the left half of the s-plane except for the possibility of one pole at the origin.

Like Tsypkin's results, the results arrived at by Jury and Lee give only sufficient conditions for asymptotic stability in-the-large of the nonlinear discrete-data system belonging to the class considered. These results are given in the form of two inequalities each of which applies to certain types of systems.

For a nonlinear discrete-data system whose nonlinearity satisfies conditions (2.1) and (2.6) the satisfaction of the following inequality on the unit circle $|z| = 1$ is a sufficient condition for asymptotic stability in-the-large of the system:

$$\mathrm{Re}\{G(z)[1+q(z-1)]\} + \frac{1}{k} - \frac{k'|q|}{2}|(z-1)G(z)|^2 \geq 0 \tag{2.8}$$

where $G(z)$ is the z-transfer function of the linear plant and q is any positive or negative[10,29] real number.

Inequality (2.8) is also applicable if the nonlinearity satisfies condition (2.7) instead of (2.6) for the case in which $k_1 = -\infty$ and $k_2 = k' < +\infty$. The constant q in this case is restricted to be non-negative. However, if $k_1 = -k'$ and $k_2 = +\infty$, then inequality (2.8) is no longer valid and the sufficient condition of system stability is the satisfaction on the circle $|z| = 1$ of the following inequality:

$$\mathrm{Re}\{G(z)[1+q\frac{z-1}{z}]\} + \frac{1}{k} - \frac{k'q}{2}|(z-1)G(z)|^2 \geq 0 \tag{2.9}$$

where q is restricted to nonnegative values.

Jury and Lee showed that Tsypkin's condition (2.2) can be obtained as a special case of their results. If no constraint is put on the slope of the nonlinearity, i.e. $k' = \infty$, then q must be set equal to zero in either (2.8) or (2.9) and in either case (2.2) results immediately.

In two later papers[11,12] Jury and Lee extended their results to apply to multivariable systems and pulse width modulated systems.

Another very interesting work in the study of stability of nonlinear discrete-data systems is the work of Epel'man[30]. He treated the problem using an entirely different approach. He investigated the system in the time domain. The system is represented by its state space equations. The nonlinear element, which may be time dependent, is assumed to satisfy conditions (2.1). By using a similarity transformation and utilizing the contraction mapping principle, he obtained a larger gain sector than that obtained by Tsypkin using his frequency domain criterion. This method, like other methods, gives only sufficient conditions for asymptotic stability in-the-large.

2.3 Bounded Input-Bounded Output Stability of Nonlinear Discrete-Data Systems

Although it has been proved that for linear systems asymptotic stability implies BIBO stability[13,14], it has been shown, by counter examples[13], that this is not, in general, true for nonlinear systems. Furthermore, a nonlinear system can exhibit local BIBO stability even though it may not be BIBO stable in a global sense.

The efforts made in the study of BIBO stability of nonlinear control systems were first directed to the case of continuous-data

systems. Sandberg[15] and Bergen et al[16], using two different approaches, showed that the satisfaction of the V.M. Popov theorem is also a sufficient condition so that the system considered is BIBO stable.

Nonlinear discrete-data systems were considered by Iwens and Bergen[17,18]. They investigated the same class of systems considered by Tsypkin and Jury and Lee. By working in the frequency domain, they showed that the satisfaction of inequalities (2.8) and (2.9) is not only sufficient for asymptotic stability in-the-large, but also sufficient for establishing BIBO stability for the classes of systems they apply to. They also showed, as a special case, that Tsypkin's criterion (2.2) also establishes BIBO stability for the class of systems it is valid for, even if the nonlinearity is time-varying, provided it satisfies condition (2.1 b) for all time.

An entirely different method, iterative in nature, was developed by Barrett[19] for investigating the problem of BIBO stability of nonlinear continuous control systems. Essentially the system response, x(t), is described in terms of the system input, y(t), in the form of the Volterra series:

$$x(t) = \int_{-\infty}^{\infty} h(t-\tau)y(\tau)d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t-\tau_1, t-\tau_2)y(\tau_1)y(\tau_2)d\tau_1 d\tau_2$$

$$+\ldots + \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} h_n(t-\tau_1, t-\tau_2,\ldots,t-\tau_n)y(\tau_1)y(\tau_2)\ldots y(\tau_n)d\tau_1 d\tau_2 \ldots d\tau_n$$

$$+\ldots. \tag{2.10}$$

where $h_n(t-\tau_1, t-\tau_2,\ldots,t-\tau_n)$ is the n^{th} order kernel and is given by:

$$h_n(t-\tau_1, t-\tau_2,\ldots,t-\tau_n) = \int_{-\infty}^{\infty} h(t-\tau)h(\tau-\tau_1)\ldots h(\tau-\tau_n)d\tau \tag{2.11}$$

In (2.11), h(t) is the impulse response of the linear part of the system. The region of convergence of the series (2.10) determines the region of BIBO stability of the system under consideration.

Christensen[20] and Christensen and Trott[21] obtained the same series solution (2.10) using functional analysis. The series is generated by a mapping which maps the space of continuous functions into itself; the input and output of the system being elements of that space. By use of the Banach fixed point theorem[25-28] the series is shown to be convergent, and consequently the system is BIBO stable, in the region where the mapping is a contraction. The advantage of this method is that the series solution generated by the mapping representing the system is unique in the region in which the mapping is a contraction.

In their work Christensen and Trott[21] included the effect of non-zero initial conditions in the series solution. By setting the input to zero they showed that the region where the mapping representing the system in this case is a contraction is also the region in which the system is asymptotically stable.

As far as nonlinear discrete-data systems are concerned, the first attempt made to use this approach for the study of these systems is due to Alper[22] although he did not consider the question of convergence of the resulting series. He represented the response of the system in terms of its input in the form of a discrete Volterra series analogous to (2.10):

$$x(m) = \sum_{k=0}^{m} h_1(k)y(m-k) + \sum_{k_1=0}^{m}\sum_{k_2=0}^{m} h_2(k_1, k_2)y(m-k_1)y(m-k_2) + \ldots + \sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} h_n(k_1,\ldots,k_n) \prod_{i=1}^{n} y(m-k_i) + \ldots \quad (2.12)$$

where $x(m)$ is the output of the system at $t=mT$, T being the sampling period, and $h_n(k_1,\ldots,k_n)$ is the n^{th} order kernel. Alper used multi-dimensional Z-transform to determine these higher order kernels.

The most extensive work done so far in the field of BIBO stability of nonlinear discrete-data systems is due to Rao[23]. He used a functional representation for the system and by utilizing the Banach fixed point theorem he obtained the region of BIBO stability. This region is the region in which the Banach fixed point theorem is satisfied. Two classes of systems were considered. The first consists of a nonlinear element located in the feedback path of an otherwise linear system. The nonlinearity is assumed to be described by:

$$\phi(x) = k_1 x + \sum_{i=2}^{N} k_i x^i \tag{2.13}$$

where k_1, k_i are constants, and N a finite integer. In the second class the nonlinearity is located in the error path rather than the feedback path.

Utilizing the same principle Rao considered BIBO stability of systems with slope restricted nonlinearities. Although the results he obtained in this case are quite conservative compared to those obtained by Jury and Lee, which also apply to BIBO stability[17,18], his method has the advantage that the actual solution of the system can be generated by a process of successive approximations, starting from the solution of the linear system as a first approximation and solving the system equation by iteration. The convergence of the solution is assured in the region in which the mapping is a contraction[23,24].

CHAPTER III

ANALYSIS OF LINEAR DISCRETE-DATA SYSTEMS WITH NON-ZERO INITIAL CONDITIONS

3.1 Introduction

In this chapter linear discrete-data systems will be considered. Two types of systems will be analysed. The first type has no hold circuit while in the second system a zero order hold device is included. Non-zero initial conditions are included in a general expression for the solution of the system. BIBO stability as well as asymptotic stability will also be discussed.

3.2 Difference Equation Representation of Linear Discrete-Data Systems[31].

While ordinary differential equations, or vector differential equations, successfully describe the dynamic behaviour of continuous-data systems, they can not describe the behaviour of discrete-data systems since they encounter derivatives of signals occuring within a given system. These derivatives are not, in general, defined for a pulse signal in a discrete-data system.

However, discrete-data systems can be conveniently represented by ordinary difference equations, or equivalently by vector difference equations. These equations relate the values of system variables at different sampling instants.

Discrete-data systems can be considered to be derived from continuous-data systems by considering the variables of such systems only at discrete instants of time. This can be achieved by the sampling

operation, Fig. 3.1. Most discrete-data systems belong to this type of systems.

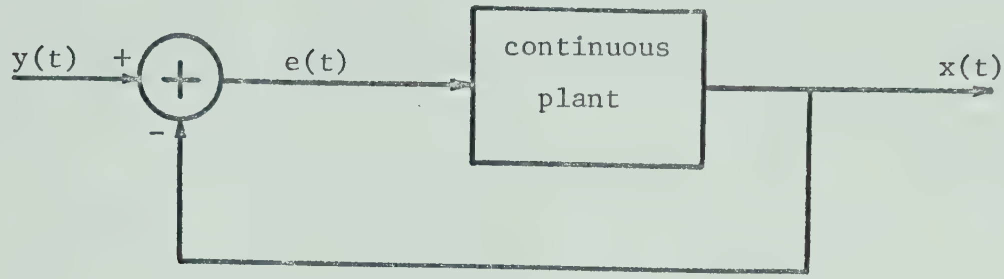

Fig. 3.1 (a)

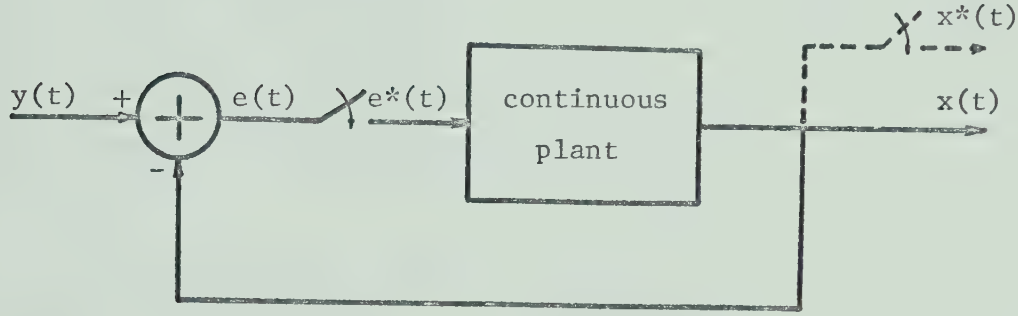

Fig. 3.1 (b)

Fig. 3.1 Development of discrete-data systems from continuous-data systems
(a) original continuous system
(b) discrete-data version

To show how to obtain the difference equation representing the discrete-data system from its continuous-data version, it is most convenient to consider a second order system as an illustrative example. However, the method is applicable to systems of arbitrary order.

Consider the continuous time-invariant driven system described by the differential equation:

$$\ddot{x}(t) + a_1\dot{x}(t) + b_1 x(t) = c_1 y(t) \tag{3.1}$$

where $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ is the output of the system and its derivatives, $y(t)$ is the input, and a_1, b_1, and c_1 are constants.

The corresponding discrete-data system is obtained by considering the quantities in (3.1) at $t=kT$, T is the sampling period, and $k=0, 1, 2,\ldots,\infty$. Furthermore, the derivatives can be approximated by the following expressions bearing in mind that T is usually small:

$$\dot{x}(t) = \frac{x(t+T) - x(t)}{T} \tag{3.2}$$

$$\ddot{x}(t) = \frac{\dot{x}(t+T) - \dot{x}(t)}{T}$$

By making use of (3.2) one gets:

$$\ddot{x}(t) = \frac{x(t+2T) - 2x(t+T) + x(t)}{T^2} \tag{3.3}$$

By substitution in (3.1) we find:

$$\frac{x[(k+2)T]-2x[(k+1)T]+x[kT]}{T^2} + a_1\frac{x[(k+1)T]-x[kT]}{T} + b_1x[kT] = c_1y[kT]$$

or

$$x[(k+2)T] + ax[(k+1)T] + bx[kT] = cy[kT] \tag{3.4}$$

where $a = a_1T - 2$

$b = b_1T^2 - a_1T + 1$

$c = c_1T^2$

Equation (3.4) is the difference equation representing the discrete version of system (3.1).

Now we consider an n^{th} order linear discrete-data system which can be described by an n^{th} order ordinary difference equation:

$$a_nx(k+n) + a_{n-1}x(k+n-1) + \ldots + a_1x(k+1) + a_ox(k) = y(k) \tag{3.5}$$

where a_i is a constant, $i=0, 1, 2,\ldots, n$; $x(k+n-i)$ stands for $x[(k+n-i)T]$, the value of the output at $t=(k+n-i)T$ and $k=0,1,2 \ldots,\infty$.

System (3.5) can be represented by a block diagram involving time delay units, adders and amplifiers. This is shown in Fig. 3.2[31].

It is clear from Fig. 3.2 that the effect of the input will appear at the output only after n sampling periods. In other words, the impulse response of system (3.5) is equal to zero for the first n sampling instants. This result can be checked by taking the Z-transform of equation (3.5). Assuming zero initial conditions one gets:

$$a_n z^n X(z) + a_{n-1} z^{n-1} X(z) + \ldots + a_1 z X(z) + a_o X(z) = Y(z)$$

which gives

$$X(z) = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_o} Y(z)$$

where X(z) and Y(z) are the Z-transforms of x(k) and y(k) respectively. The system z-transfer function is thus given by:

$$G(z) = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_o} \qquad (3.6)$$

The discrete impulse response can be obtained by inverse Z-transforming (3.6). Using the power series method it is evident that the value of the unit impulse response at $t=0, T, 2T, \ldots, (n-1)T$ is:

$$g(0) = g(T) = \ldots\ldots. = g((n-1)T) = 0 \qquad (3.7)$$

which confirms the result arrived at from block diagram considerations.

At this point it is very important to note that the type of systems considered in this section has no hold device. Systems with zero order hold devices will be considered at the end of this chapter.

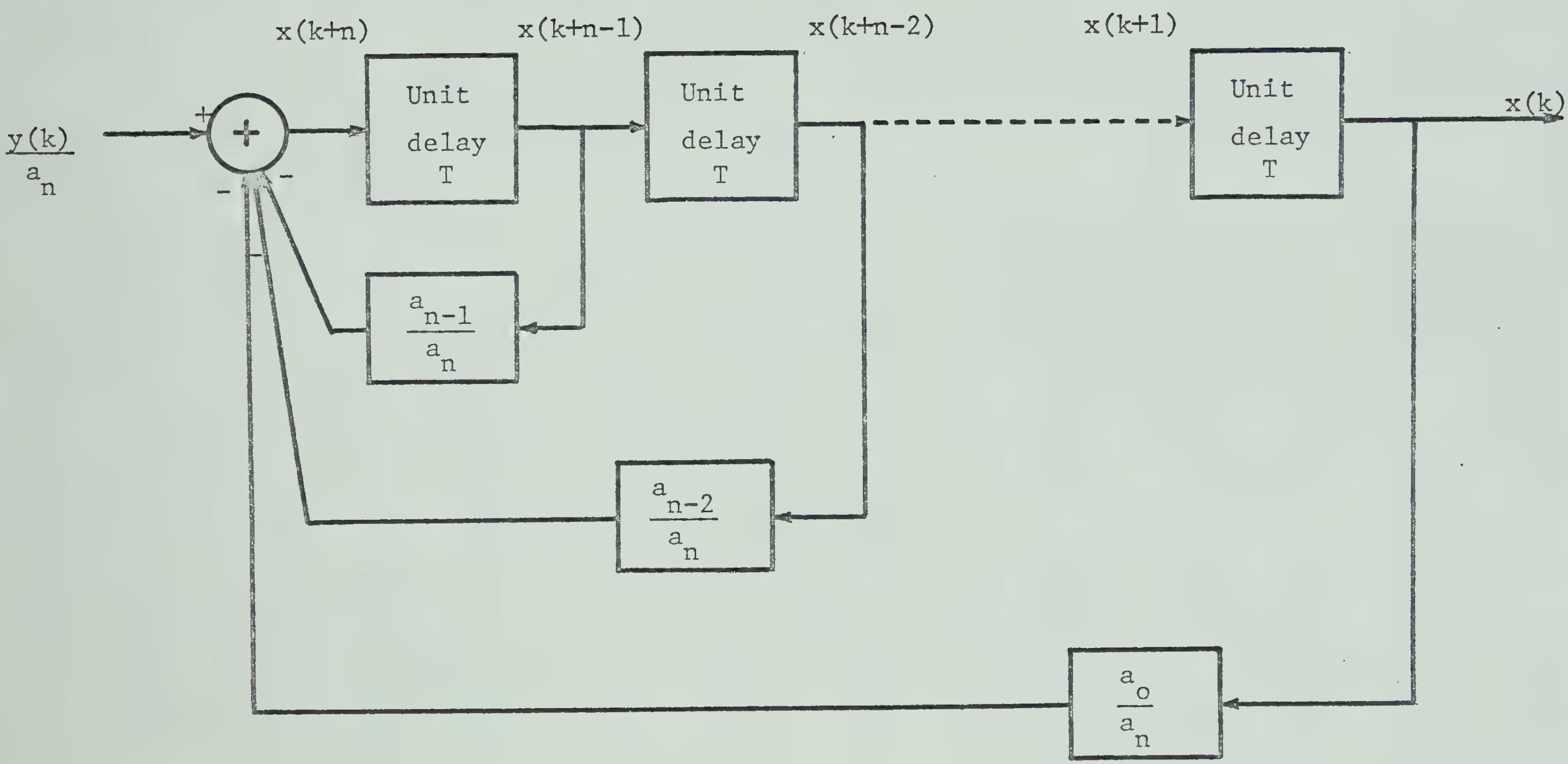

Fig. 3.2 Block Diagram of System (3.5)

3.3 Response of Linear Discrete-Data Systems With Non-Zero Initial Conditions[32].

In this section we consider a linear discrete-data system with no hold device. The system is assumed to be time-invariant and of arbitrary order. The difference equation representing the system is given by equation (3.5), viz.:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + a_1 x(k+1) + a_o x(k) = y(k) \qquad (3.8)$$

where $x(k+n-i)$ is the value of the system response at $t=(k+n-i)T$; T is the sampling period; $y(k)$ is a bounded driving sampled function; a_i is a constant, $i=0, 1, \ldots, n$; n is the order of the system, and $k=0, 1, 2, \ldots, \infty$, representing the sampling instants. It is assumed that $y(k)$ is zero for negative k.

By taking the Z-transform of equation (3.8) one gets:

$$\begin{aligned} & a_n z^n [X(z) - x(0) - x(1)z^{-1} \ . \ . \ . -x(n-1)z^{n-1}] \\ & \quad + a_{n-1} z^{n-1} [X(z) - x(0) - x(1)z^{-1} \ . \ . \ . -x(n-2)z^{n-2}] \\ & \quad + \ . \ . \ . \ . \ . \\ & \quad + a_1 z[X(z) - x(0)] + a_o X(z) = Y(z) \end{aligned}$$

Rearranging we find

$$\begin{aligned} & [a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_o] \, X(z) \\ & \quad = Y(z) + [a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z] \, x(0) \\ & \qquad + [a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_2 z] \, x(1) \\ & \qquad + \ . \ . \ . \ . \ . + [a_n z^2 + a_{n-1} z] \, x(n-2) \\ & \qquad + [a_n z] \, x(n-1) \end{aligned} \qquad (3.9)$$

where $x(0), x(1), \ldots, x(n-1)$ are the initial conditions.

Now we define the initial vector $\underline{x}(o)$ as:

$$\begin{aligned} \underline{x}(0) &= [x(0) \quad x(1) \quad . \ . \ . \ . \ . \ x(n-1)]^T \\ &= [x_1(0) \quad x_2(0) \ . \ . \ . \ . \ . \ x_n(0)]^T \end{aligned} \qquad (3.10)$$

Also let L(z) define the delay operator multiplying X(z) in (3.9). It is given by:

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_o \tag{3.11}$$

Making use of (3.10) and (3.11), equation (3.9) becomes:

$$\begin{aligned} X(z) = \frac{1}{L(z)} [&z(a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_1)\, x_1(0) \\ &+ z(a_n z^{n-2} + a_{n-1} z^{n-3} + \ldots + a_2)\, x_2(0) \\ &+ \ldots . + z(a_n z + a_{n-1})\, x_{n-1}(0) \\ &+ z(a_n)\, x_n(0) + Y(z)] \end{aligned} \tag{3.12}$$

From (3.12) it can be seen that the coefficient multiplying $x_i(0)$, i=1, 2, ..., n, can be written as:

$$B_i(z) = \frac{1}{L(z)}\, z[a_n z^{n-i} + a_{n-1} z^{n-i-1} + \ldots + a_i] \tag{3.13}$$

Substituting (3.13) into (3.12) one gets:

$$\begin{aligned} X(z) &= \sum_{i=1}^{n} B_i(z)\, x_i(0) + \frac{1}{L(z)}\, Y(z) \\ &= \sum_{i=1}^{n} B_i(z)\, x_i(0) + G(z)\, Y(z) \end{aligned} \tag{3.14}$$

$G(z) = \frac{1}{L(z)}$ defines the z-transfer function of system (3.8).

Now inverse z-transforming (3.14) we get:

$$x(k) = \sum_{i=1}^{n} b_i(k)\, x_i(0) + \sum_{j=0}^{k} g(j)\, y(k-j) \tag{3.15}$$

where

$$g(j) = Z^{-1}[G(z)] = \frac{1}{2\pi j} \oint_{\Gamma_1} G(z)\, z^{j-1}\, dz \tag{3.16*}$$

and

$$b_i(k) = Z^{-1}[B_i(z)] = \frac{1}{2\pi j} \oint_{\Gamma_2} B_i(z)\, z^{k-1}\, dz \tag{3.17*}$$

---

* See the appendix.

Defining $< \underline{u}, \underline{v} >$ as the inner product of the two vectors $\underline{u}$ and $\underline{v}$, equation (3.15) can be written as:

$$x(k) = < \underline{b}(k), \underline{x}(0) > + \sum_{j=0}^{k} g(j)\, y(k-j) \tag{3.18}$$

where

$$\underline{b}(k) = [b_1(k)\;\; b_2(k)\; \ldots\ldots\; b_n(k)]^T \tag{3.19}$$

Equation (3.18) gives the response of the system (3.8) at $t=kT$, $k=0, 1, \ldots, \infty$.

The second term in (3.18) is the convolution summation between the sampled input, $y^*(t)$, and the discrete impulse response of the system, $g^*(t)$. Here we have:

$$\sum_{j=0}^{k} g(j)\, y(k-j) = \sum_{j=0}^{k} g(k-j)\, y(j) \tag{3.20}$$

Substituting (3.20) into (3.18) we get:

$$x(k) = < \underline{b}(k), \underline{x}(0) > + \sum_{j=0}^{k} g(k-j)\, y(j) \tag{3.21}$$

The discrete response of the system is then given by:

$$x^*(t) = \sum_{k=0}^{\infty} [<\underline{b}(k), \underline{x}(0)> + \sum_{j=0}^{k} g(j)\, y(k-j)]\, \delta(t-kT) \tag{3.22a}$$

$$= \sum_{k=0}^{\infty} [<\underline{b}(k), \underline{x}(0)> + \sum_{j=0}^{k} g(k-j)\, y(j)]\, \delta(t-kT) \tag{3.22b}$$

where $\delta(t-kT)$ is given by:

$$\delta(t-kT) \begin{cases} = 1 & \text{for } t=kT \\ = 0 & \text{for } t\neq kT \end{cases} \tag{3.23}$$

3.4 The Functional Representation[33,34].

The dynamic behaviour of any system, linear or nonlinear, is usually described by a differential or integral equation, or their discrete version if the system considered is a discrete-data system. However, only in the case of linear systems can one, in general, solve the system equation explicitly with relative ease. In contrast, in the case of nonlinear systems, it is very difficult, and usually impossible, to solve the system equations explicitly. For this reason it is very advantageous to use functional analysis.

This approach utilizes as its tools the concept of abstract spaces and operations on such spaces. The input and output elements of the system are identified as elements in these spaces with the system considered as a mapping, or a transformation, of the input space into the output space.

An abstract Banach space is suitable for studying discrete-data systems. One uses such a space when applying the contraction mapping theorem to system analysis. The following definitions will clarify what is meant by a Banach space.

Definition 1: A collection of elements together with a certain structure of relations between elements or of rules of manipulation and combination, the whole supporting a mathematical development, is often called a space.

Definition 2: A space X is said to be a linear, vector, space if addition and scalar multiplication are defined on X satisfying the commutative, associative, and distributive laws. The scalar multiplication is related to some associated field of scalars, the real or the complex field, usually denoted by $\mathcal{F}$.

Definition 3: A space X is said to be a metric space, if with every pair of points x_1, x_2 of X is associated a distance function or metric $d(x_1, x_2)$ such that:

(a) $d(x_1, x_2) > 0$, if $x_1 \neq x_2$

(b) $d(x_1, x_2) = 0$, if and only if $x_1 = x_2$

(c) $d(x_1, x_2) = d(x_2, x_1)$

(d) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$ for every $x_3 \varepsilon X$

Definition 4: A norm on a linear space X is a real valued function, whose value at x is denoted by $||x||$, with the properties:

(a) $||x|| \geq 0$

(b) $||x|| \neq 0$, if $x \neq 0$

(c) $||x_1 + x_2|| \leq ||x_1|| + ||x_2||$

(d) $||ax|| = |a| \; ||x||$, for all $a \varepsilon \mathcal{F}$

The space X with this norm is called a normed linear space.

Definition 5: A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space is called a Cauchy sequence if $d(x_n, x_m) \to 0$ as m and $n \to \infty$.

Definition 6: A metric space X is said to be complete if every Cauchy sequence of elements of X has a limit in X.

Definition 7: A Banach space is a normed linear space which is also a complete metric space with respect to the metric $d(x_1, x_2) = ||x_1 - x_2||$ induced by the norm.

Definition 8: Let X and Y be two linear spaces with the same field of scalars and let A: $X \to Y$ be a function defined on X with values in Y. A is called a linear operator from X into Y if it has the following properties:

(a) $A(x_1 + x_2) = A(x_1) + A(x_2)$ for any $x_1, x_2 \varepsilon X$.

(b) $A(ax) = aA(x)$ for any $x \varepsilon X$ and any scalar a.

Now to obtain the functional representation of system (3.8) we begin with either equation (3.22a) or (3.22b). Expanding equation (3.22a) we get:

$$x^*(t) = \sum_{k=0}^{\infty} < \underline{b}(k), \underline{x}(0) > \delta(t - kT) + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)\ y(k-j)]\ \delta(t-kT)$$

$$= \sum_{k=0}^{\infty} < \underline{b}(k)\ \delta(t - kT), \underline{x}(0) > + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)\ y(k-j)]\ \delta(t-kT) \quad (3.24)$$

Now

$$\sum_{k=0}^{\infty} < \underline{b}(k)\ \delta(t-kT), \underline{x}(0) > = \sum_{k=0}^{\infty} \sum_{i=1}^{n} b_i(k)\ \delta(t-kT)\ x_i(0)$$

$$= \sum_{i=1}^{n} x_i(0) \sum_{k=0}^{\infty} b_i(k)\ \delta(t-kT)$$

$$= \sum_{i=1}^{n} x_i(0)\ b_i^*(t) = < \underline{b}^*(t), \underline{x}(0) >$$

Substituting this back into (3.24) one gets:

$$x^*(t) = < \underline{b}^*(t), \underline{x}(0) > + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)\ y(k-j)]\ \delta(t-kT)$$

$$= f^*(t) + \beta^* y^*(t) \quad (3.25)$$

where $f^*(t) = < \underline{b}^*(t), \underline{x}(0) >$ (3.26)

and β^* is an operator defined by

$$\beta^* y^*(t) = \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)\ y(k-j)]\ \delta(t-kT) \quad (3.27a)$$

or, starting with equation (3.22b),

$$\beta^* y^*(t) = \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(k-j)\, y(j)]\, \delta(t-kT) \tag{3.27b}$$

It is easy to prove that β^* is a linear operator.

Now the input $y^*(t)$ and the output $x^*(t)$ are real valued functions of time defined only at the discrete instants $t = kT$, $k = 0, 1, 2, \ldots, \infty$. They are thus sequences in a sequence space, say S, which is a linear space. Addition and scalar multiplication in this space are as follows:

Addition: If $x^* = \{x_n\}_{n=0}^{\infty}$ and $y^* = \{y_n\}_{n=0}^{\infty}$, then $z^* = x^* + y^*$, where $z^* = \{z_n\}_{n=0}^{\infty}$, $z_n = x_n + y_n$.

Multiplication: $ax^* = a\{x_n\}_{n=0}^{\infty} = \{ax_n\}_{n=0}^{\infty}$, $a \varepsilon \mathcal{F}$. Here $\mathcal{F}$ is the field of real scalars.

If furthermore, $y^*(t)$ and $x^*(t)$ are bounded, they belong to the space of bounded sequences, say F, which is a subspace of S. F is a Banach space if we define the norm:

$$||x^*(t)|| = \sup_{-\infty<t<\infty} |x^*(t)| \text{ , } x^* \varepsilon F. \tag{3.28}$$

Now equation (3.25) can be written as

$$x^*(t) = A^*\, y^*(t) \tag{3.29}$$

where A^* is an operator defined by (3.25). A^* is a mapping of S into itself in the general case, while for the special case where $y^*(t)$ and $x^*(t)$ are bounded, A^* maps F into F.

3.5 Stability of Linear Discrete-Data Systems[35].

In this section asymptotic stability as well as BIBO stability of the linear discrete-data system (3.8) will be described. As a preliminary development we will discuss further the coefficients $B_i(z)$, and consequently $b_i(k)$, $i = 1, 2, \ldots, n$, defined by equations (3.13) and (3.17) respectively.

$B_i(z)$ can be rewritten as:

$$B_i(z) = \frac{1}{L(z)} [a_n z^{n-i+1} + a_{n-1} z^{n-i} + \ldots + a_i z]$$

$$= G(z)[a_n z^{n-i+1} + a_{n-1} z^{n-i} + \ldots + a_i z] \tag{3.30}$$

Now, according to (3.7) we have:

$$Z[g^*(t+mT)] = z^m G(z) - z^m \sum_{j=0}^{m-1} g(jT)\, z^{-j}$$

$$= z^m G(z), \quad m = 1, 2, \ldots, n. \tag{3.31}$$

or

$$g^*(t+mT) = Z^{-1}[z^m G(z)], \; m = 1, 2, \ldots, n. \tag{3.32}$$

Inverse z-transforming (3.30) and utilizing (3.31) and (3.32) we find:

$$b_i^*(t) = a_n g^*[t+(n-i+1)T] + a_{n-1} g^*[t+(n-i)T]$$
$$+ \ldots + a_i g^*(t+T) \tag{3.33}$$

or

$$b_i(k) = a_n g(k+n-i+1) + a_{n-1} g(k+n-i) + \ldots + a_i g(k+1) \tag{3.34}$$

where $i = 1, 2, \ldots, n$.

Substituting (3.34) into (3.15) we find:

$$x(k) = \sum_{i=1}^{n} [a_n g(k+n-i+1) + a_{n-1} g(k+n-i) + \ldots + a_i g(k+1)]\, x_i(0)$$
$$+ \sum_{j=0}^{k} g(j)\, y(k-j) \tag{3.35}$$

3.5.1 BIBO Stability.

By taking the absolute value of both sides of equation (3.35) one gets:

$$|x(k)| \leq \sum_{i=1}^{n} |a_n g(k+n-i+1) + a_{n-1} g(k+n-i) + \ldots + a_i g(k+1)|\,|x_i(0)|$$

$$+ \sum_{j=0}^{k} |g(j)||y(k-j)|$$

$$\leq \sum_{i=1}^{n} \max_{i\leq j\leq n} |a_j| \; |g(k+n-i+1) + g(k+n-i) + \ldots + g(k+1)| \; |x_i(0)|$$

$$+ \sup_{0\leq k<\infty} |y(k)| \sum_{j=0}^{\infty} |g(j)|$$

$$\leq \sum_{i=1}^{n} [\max_{i\leq j\leq n} |a_j|][(n-i+1) \sup_{0\leq j<\infty} |g(j)|]|x_i(0)| + ||y^*(t)|| \sum_{j=0}^{\infty} |g(j)|$$

$$\leq \max_{1\leq j\leq n} |a_j| . \max_{1\leq i\leq n} |x_i(0)| . \sup_{0\leq j<\infty} |g(j)| . n^2$$

$$+ ||y^*(t)|| \sum_{j=0}^{\infty} |g(j)| \tag{3.36}$$

Since the input is bounded and $||y^*(t)||$ is finite, the R.H.S. of inequality (3.36) is bounded if :

$$\sum_{j=0}^{\infty} |g(j)| < \infty \tag{3.37}$$

Thus since (3.36) is true for all k, (3.37) is a sufficient condition that $||x^*(t)||$ is bounded. Also it has been proved that (3.37) is also a necessary condition for BIBO stability[35].

Note that (3.37) implies that:

$$\lim_{j\to\infty} |g(j)| = 0 \tag{3.38}$$

3.5.2 Asymptotic Stability

For the purpose of studying asymptotic stability, we let the input of the system $y^*(t)$, equal to zero in (3.35) to get:

$$x(k) = \sum_{i=1}^{n} [a_n g(k+n-i+1) + a_{n-1} g(k+n-i) + \ldots + a_i g(k+1)] x_i(0) \tag{3.39}$$

Now asymptotic stability requires:

$$\lim_{k\to\infty} |x(k)| = 0 \qquad (3.40)$$

From (3.39) we find:

$$|x(k)| \leq \sum_{i=1}^{n} \{|a_n||g(k+n-i+1)| + |a_{n-1}||g(k+n-i)| +\ldots+|a_i||g(k+1)|\}|x_i(0)| \qquad (3.41)$$

Clearly, since a_i and $x_i(0)$, i=1, 2,..., n, are finite and are not all zero, hence if (3.37) holds, then so does (3.38) and the R.H.S. of (4.41) will tend to zero as k tends to infinity. Therefore (3.37) is also a necessary and sufficient condition for asymptotic stability of linear discrete-data systems.

3.6 Systems With Zero Order Hold.

For the purpose of investigating linear discrete-data systems which include zero order hold (ZOH) devices, it is again preferable to represent the system by a difference equation. However, in this case the approach of section 3.2 can not be used since the differential equation representing the plant together with the ZOH is rather complicated. It seems logical that to determine the required difference equation, one can make use of the z-transfer function of the whole system[14,35].

To illustrate this we consider a system whose overall z-transfer function together with the ZOH is given as a ratio of two polynomials in z. The number of its zeros is at least one less than the number of its poles. This is usually the case in most physical systems. The impulse response of such a system will be zero for t=0. In other words g(kT) = 0 for k=0.

Mathematically:

$$G(z) = \frac{X(z)}{Y(z)} = \frac{c_m z^m + c_{m-1} z^{m-1} + \ldots + c_o}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_o} \tag{3.42}$$

where X(z) and Y(z) are the z-transforms of the output and input respectively, and $m < n$.

Since the z-transfer function G(z) is independent of initial conditions, cross multiplying (3.42), inverse z-transforming, and taking into consideration relation (3.32), one gets:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + a_o x(k)$$
$$= c_m y(k+m) + c_{m-1} y(k+m-1) + \ldots + c_o y(k) \tag{3.43}$$

Equation (3.43) is the difference equation representing the system.

Now to obtain the response of the system when initial conditions are not zero, we take the z-transform of equation (3.43). Hence[35,36]:

$$X(z) = G(z)\ Y(z) + \sum_{i=1}^{n} B_i(z)\ x_i(0) - \sum_{i=1}^{m} F_i(z)\ y_i(0) \tag{3.44}$$

where the initial vectors $\underline{x}(0)$ and $\underline{y}(0)$ are given by:

$$\underline{x}(0) \equiv [x(0)\ x(1)\ .\ .\ .\ .\ x(n-1)]^T$$
$$= [x_1(0)\ x_2(0)\ .\ .\ .\ .\ x_n(0)]^T \tag{3.45}$$

and

$$\underline{y}(0) \equiv [y(0)\ y(1)\ .\ .\ .\ .\ y(m-1)]^T$$
$$= [y_1(0)\ y_2(0)\ .\ .\ .\ .\ y_m(0)]^T \tag{3.46}$$

$B_i(z)$ is given by (3.13), and $F_i(z)$ is given by:

$$F_i(z) = \frac{z}{L(z)} [c_m z^{m-i} + c_{m-1} z^{m-i-1} + \ldots + c_i] \tag{3.47}$$

L(z) is given by (3.11).

By following the same procedure as in section 3.3 we get:

$$x(k) = < \underline{b}(k), \underline{x}(0)> - <\underline{f}(k), \underline{y}(0) > + \sum_{j=0}^{k} g(j)y(k-j) \qquad (3.48)$$

where $\underline{b}(k)$, $\underline{f}(k)$ and $g(j)$ are defined by relations similar to (3.17), (3.19) and (3.16).

The discrete response of system (3.43) is then given by:

$$x^*(t) = \sum_{k=0}^{\infty} [<\underline{b}(k),\underline{x}(0)> - <\underline{f}(k),\underline{y}(0)> + \sum_{j=0}^{k} g(j)\ y(k-j)]\delta(t-kT) \qquad (3.49a)$$

$$= \sum_{k=0}^{\infty} [<\underline{b}(k),x(0)> - <\underline{f}(k),\underline{y}(0)> + \sum_{j=0}^{k} g(k-j)y(j)]\delta(t-kT) \qquad (3.49b)$$

where $\delta(t-kT)$ is the delta function defined by (3.23).

As far as system stability is concerned, an argument like that of section 3.5 can be used to show that conditions (3.37) and (3.38) are also applicable for systems with ZOH devices.

CHAPTER IV

ANALYSIS OF NONLINEAR DISCRETE-DATA SYSTEMS WITH NON-ZERO INITIAL CONDITIONS

4.1 Introduction

In this chapter we will consider time invariant, nonlinear discrete-data systems. It will be shown that the response of these systems can be represented in the form of the discrete Volterra series. The series is developed by solving the system equation by iteration. The convergence of the series, and consequently system stability, is investigated by the aid of the Banach fixed point theorem.

Systems without hold circuits will be discussed first, and then the results will be extended to systems including ZOH devices.

4.2 Description of the System

The configuration of the nonlinear discrete-data system under investigation is shown in Fig. 4.1. It is a single-loop negative feedback system with one nonlinear element N located in the feedback path. The forward path consists of a sampler followed by a linear plant G_o. The sampling is uniform and of period T. $x(t)$, $y(t)$ and $e(t)$ denote the output, input and error signals respectively; they are continuous real valued functions of time. $x^*(t)$, $y^*(t)$ and $e^*(t)$ are the corresponding sampled functions.

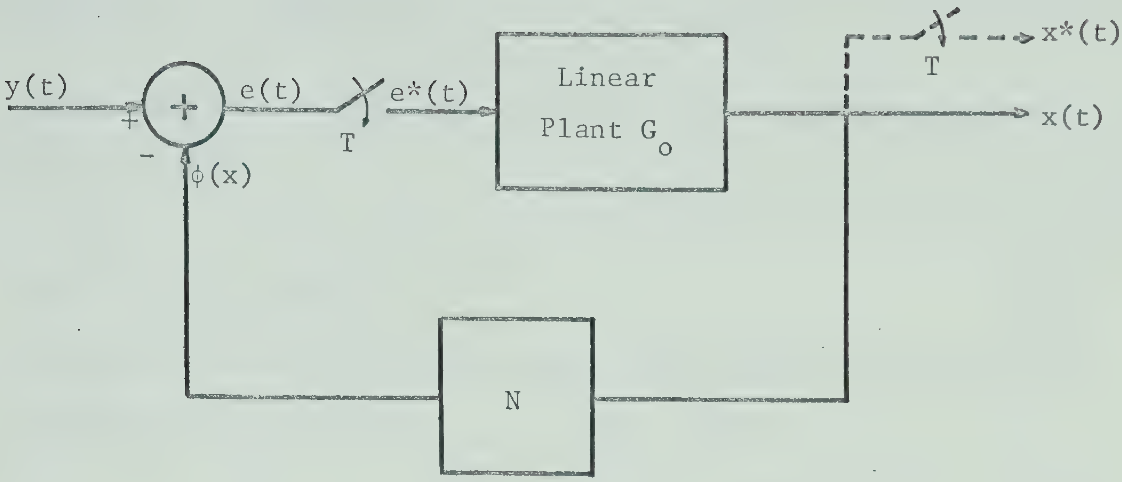

Fig. 4.1 Nonlinear Discrete-Data System Under Investigation

The nonlinear gain function $\phi(x)$ is assumed to be continuous in its argument, memoryless, and further satisfies the condition $\phi(0) = 0$. These assumptions allow its representation in the form:

$$\phi(x) = \gamma_1 x + \sum_{i=2}^{M} \gamma_i x^i \tag{4.1}$$

$$= \gamma_1 x + \phi_1(x) \tag{4.2}$$

where γ_1, γ_i are constants, and M a finite integer.

It is assumed further that the corresponding linearized system obtained by neglecting $\phi_1(x)$ is BIBO stable.

4.3 Response of Nonlinear Discrete-Data Systems Containing No Hold Devices

The system considered in this section has no hold device included in the linear plant G_o. The forward path can thus be represented by a difference equation of the form (3.8):

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + a_o x(k) = e(k) \tag{4.3}$$

where x(k) and e(k) are the output and the error signals measured at t=kT respectively.

But

$$e(k) = y(k) - \phi(x(k)) \tag{4.4}$$

then, substituting (4.4) into (4.3) and making use of (4.2) one gets:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + (a_o+\gamma_1)\, x(k) = y(k) - \phi_1(x(k)) \tag{4.5}$$

Equation (4.5) represents the nonlinear discrete-data system under investigation.

Note that if $\phi_1(x(k))$ is set equal to zero in (4.5), we get:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + (a_o+\gamma_1)\, x(k) = y(k) \tag{4.6}$$

which is the equation of the linearized system corresponding to the nonlinear system (4.5). The solution of (4.6) is thus given by:

$$v^*(t) = \sum_{k=0}^{\infty} [< \underline{b}(k), \underline{x}(0) > + \sum_{j=0}^{k} g(j)\, y(k-j)]\, \delta(t-kT) \tag{4.7}$$

or

$$v^*(t) = f^*(t) + \beta^* y^*(t) \tag{4.8}$$

where

$$f^*(t) = < \underline{b}^*(t), \underline{x}(0) > \tag{4.9}$$

β^* is the convolution summation operator. It is defined by:

$$\beta^* y^*(t) = \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)\, y(k-j)]\, \delta(t-kT) \tag{4.10}$$

$$g(j) = Z^{-1}[G(z)] = Z^{-1}[\frac{1}{L(z)}] \tag{4.11}$$

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + (a_o+\gamma_1) \tag{4.12}$$

The vector $\underline{b}(k)$ is defined by equations (3.19), (3.17) and (3.13). Also the initial vector $\underline{x}(0)$ is defined by equation (3.10). $g(j)$ is the impulse response of the linearized system evaluated at $t=jT$.

Now, we consider the nonlinear system (4.5), and we let

$$e_1(k) = y(k) - \phi_1(x(k)) \tag{4.13}$$

Then, equation (4.5) becomes:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \dots + (a_o+\gamma_1)\, x(k) = e_1(k) \tag{4.14}$$

which has the same form as equation (4.6) with $e_1(k)$ taking the place of $y(k)$. Thus (4.14) has a solution:

$$x^*(t) = \sum_{k=0}^{\infty} [< \underline{b}(k), \underline{x}(0) > + \sum_{j=0}^{k} g(j)\, e_1(k-j)]\, \delta(t-kT) \tag{4.15}$$

or

$$x^*(t) = f^*(t) + \beta^* e_1{}^*(t) \tag{4.16}$$

Since β^* is a linear operator, expanding (4.16) using (4.13) we get:

$$x^*(t) = f^*(t) + \beta^* y^*(t) - \beta^*\phi_1^*(x(k)) \tag{4.17}$$

$$= v^*(t) - \beta^*\phi_1^*(x(k)) \tag{4.18}$$

where $v^*(t)$ is the solution of the corresponding linearized system given by equation (4.8).

Now, we substitute in (4.17) and (4.18) with the value of $\phi_1^*(x(k))$ obtained from equations (4.1) and (4.2) to get:

$$x^*(t) = f^*(t) + \beta^* y^*(t) - \beta^*[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)] \tag{4.19}$$

$$= v^*(t) - \beta^*[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)] \tag{4.20}$$

With the input $y^*(t)$ fixed, equation (4.19) or (4.20) defines a relation of the form:

$$x^*(t) = U\, x^*(t) \tag{4.21}$$

where U is a nonlinear operator defined by:

$$U\, x^*(t) = f^*(t) + \beta^* y^*(t) - \beta^*\Big[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)\Big] \tag{4.22}$$

$$= v^*(t) - \beta^*\Big[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)\Big] \tag{4.23}$$

Equation (4.20) can be solved for $x^*(t)$ by a method of successive approximations starting with the solution of the linearized system $v^*(t)$ as a first approximation. This is carried out as follows.

Without loss of generality, we assume that the nonlinearity is given by:

$$\phi(x(k)) = x(k) + \varepsilon\, x^3(k) \tag{4.24}$$

which is obtained from (4.1) by letting $\gamma_1=1$, $\gamma_3=\varepsilon$, $\varepsilon>0$, and $\gamma_i=0$, $i=2, 4, 5, \ldots, M$.

The nonlinear difference equation representing the nonlinear system in this case is obtained by substituting (4.24) into (4.5) to get:

$$a_n x(k+n) + a_{n-1} x(k+n-1) + \ldots + (a_o+1)x(k) = y(k) - \varepsilon x^3(k) \tag{4.25}$$

The solution of the system is, thus:

$$x^*(t) = f^*(t) + \beta^* y^*(t) - \varepsilon\beta^*\, x^{3^*}(t) \tag{4.26}$$

$$= v^*(t) - \varepsilon\beta^*\, x^{3^*}(t) \tag{4.27}$$

The operator U is also given by:

$$Ux^*(t) = f^*(t) + \beta^* y^*(t) - \varepsilon\beta^*\, x^{3^*}(t) \tag{4.28}$$

$$= v^*(t) - \varepsilon\beta^*\, x^{3^*}(t) \tag{4.29}$$

To carry out the process of successive approximations, it is more convenient to rewrite equation (4.27) for the output at the end of the k^{th} sampling period, thus;

$$x(k) = v(k) - \varepsilon\beta\, x^3(k) \tag{4.30}$$

where β represents the convolution summation operator for that sampling instant. β is defined by the relation:

$$\beta u(k) = \sum_{j=0}^{k} g(k-j)\ u(j) \tag{4.31}$$

where $g(n)$ represents the impulse response of the linearized system at the end of the n^{th} sampling period. Utilizing (4.31) in (4.30) one gets:

$$x(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)\ x^3(k) \tag{4.32}$$

Now, the solution $v(k)$ of the linearized system will be the first approximation $x_o(k)$, i.e.:

$$x_o(k) = v(k) \tag{4.33}$$

Substituting (4.33) into the R.H.S. of (4.32) we obtain the second approximation for $x(k)$, viz. $x_1(k)$:

$$x_1(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)\ v^3(k) \tag{4.34}$$

Further substitution of (4.34) into the R.H.S. of (4.32) results in the third approximation:

$$x_2(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)[v(j) - \varepsilon \sum_{j_1=0}^{j} g(j-j_1)\ v^3(j_1)]^3 \tag{4.35}$$

The second term in (4.35) can be expanded in the following way:

$$\varepsilon \sum_{j=0}^{k} g(k-j)[v(j) - \varepsilon \sum_{j_1=0}^{j} g(j-j_1)\ v^3(j_1)]^3$$

$$= \varepsilon \sum_{j=0}^{k} g(k-j)\ \{v^3(j) - 3\varepsilon\ v^2(j) \sum_{j_1=0}^{j} g(j-j_1)\ v^3(j_1)$$

$$+ 3\varepsilon^2 v(j)[\sum_{j_1=0}^{j} g(j-j_1)\, v^3(j_1)]^2$$

$$-\varepsilon^3[\sum_{j_1=0}^{j} g(j-j_1)\, v^3(j_1)]^3\}$$

$$= \varepsilon \sum_{j=0}^{k} g(k-j)\, v^3(j) - 3\varepsilon^2 \sum_{j=0}^{k} \sum_{j_1=0}^{j} g(k-j)g(j-j_1)\, v^2(j)\, v^3(j_1)$$

$$+ 3\varepsilon^3 \sum_{j=0}^{k} \sum_{j_1=0}^{j} \sum_{j_2=0}^{j} g(k-j)g(j-j_1)g(j-j_2)\, v(j)\, v^3(j_1)\, v^3(j_2)$$

$$- \varepsilon^4 \sum_{j=0}^{k} \sum_{j_1=0}^{j} \sum_{j_2=0}^{j} \sum_{j_3=0}^{j} [g(k-j)g(j-j_1)g(j-j_2)g(j-j_3)v^3(j_1)v^3(j_2)\, v^3(j_3)] \tag{4.36}$$

But since $g(-n) = 0$ for all positive integers n, we can write:

$$\sum_{j=0}^{k} \sum_{j_1=0}^{j} g(k-j)g(j-j_1)\, v^2(j)\, v^3(j_1)$$

$$= \sum_{j=0}^{k} \sum_{j_1=0}^{k} g(k-j)g(j-j_1)\, v^2(j)\, v^3(j_1) \tag{4.37}$$

Similarly for the other terms.

Substituting (4.36) into (4.35) making use of (4.37) we obtain:

$$x_2(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)\, v^3(j) + 3\varepsilon^2 \sum_{j=0}^{k} \sum_{j_1=0}^{k} g(k-j)g(j-j_1)v^2(j)v^3(j_1)$$

$$-3\varepsilon^3 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} g(k-j)g(j-j_1)g(j-j_2)v(j)v^3(j_1)v^3(j_2)$$

$$+ \varepsilon^4 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \sum_{j_3=0}^{k} g(k-j)g(j-j_1)g(j-j_2)g(j-j_3)v^3(j_1)v^3(j_2)v^3(j_3) \tag{4.38}$$

To obtain higher order approximations we must then substitute equation (4.38) into the R.H.S. of (4.32). Continuing this process, we obtain the solution for the nonlinear discrete-data system (4.25) in the form of the infinite series:

$$x(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)\, v^3(j)$$
$$+ 3\varepsilon^2 \sum_{j=0}^{k} \sum_{j_1=0}^{k} g(k-j)g(j-j_1)v^2(j)v^3(j_1)$$
$$- 3\varepsilon^3 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} g(k-j)g(j-j_1)g(j-j_2)v(j)v^3(j_1)v^3(j_2)$$
$$+ \varepsilon^4 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \sum_{j_3=0}^{k} g(k-j)g(j-j_1)g(j-j_2)g(j-j_3)v^3(j_1)v^3(j_2)v^3(j_3)$$
$$+ \ldots + \ldots \tag{4.39}$$

Note here that if the system is subject to zero initial conditions, then $f(k) = 0$, and the solution of the linearized system will be given by:

$$v(k) = \sum_{j=0}^{k} g(k-j)y(j) \tag{4.40}$$

Substituting (4.40) into (4.39), the series solution of the nonlinear system will then reduce to the form obtained by Rao[23]:

$$x(k) = \sum_{j=0}^{k} g(k-j)y(j) - \varepsilon \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \sum_{j_3=0}^{k} g_3 y(j_1)y(j_2)y(j_3)$$
$$+ \ldots + \ldots \tag{4.41}$$

where g_3 is defined by:

$$g_3 = \sum_{j=0}^{k} g(k-j)g(j-j_1)g(j-j_2)g(j-j_3) \tag{4.42}$$

Now by taking the absolute value of both sides of (4.39), we get:

$$|x(k)| \le |v(k)| + |\varepsilon| \sum_{j=0}^{k} |g(k-j)||v(j)|^3$$

$$+ 3|\varepsilon|^2 \sum_{j=0}^{k} \sum_{j_1=0}^{k} |g(k-j)||g(j-j_1)||v(j)|^2|v(j_1)|^3$$

$$+ 3|\varepsilon|^3 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} |g(k-j)||g(j-j_1)||g(j-j_2)||v(j)||v(j_1)|^3|v(j_2)|^3$$

$$+ \; . \; . \; . \; .$$

$$\le \sup_{0\le k<\infty} |v(k)| + |\varepsilon|[\sup_{0\le k<\infty} |v(k)|]^3 \sum_{j=0}^{k} |g(k-j)|$$

$$+ 3|\varepsilon|^2[\sup_{0\le k<\infty} |v(k)|]^5 \sum_{j=0}^{k} \sum_{j_1=0}^{k} |g(k-j)||g(j-j_1)|$$

$$+ 3|\varepsilon|^3[\sup_{0\le k<\infty} |v(k)|]^7 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} |g(k-j)||g(j-j_1)||g(j-j_2)|$$

$$+ \; . \; . \; . \; . \qquad (4.43)$$

Since (4.43) is true for all k, it can be written in the form:

$$||x^*(t)|| \le ||v^*(t)|| + |\varepsilon| \; ||g_3|| \; ||v^*(t)||^3 + 3|\varepsilon|^2||g_5|| \; ||v^*(t)||^5$$

$$+ 3|\varepsilon|^3||g_7|| \; ||v^*(t)||^7 + \; . \; . \; . \qquad (4.44)$$

or

$$||x^*(t)|| \le ||v^*(t)|| + \sum_{n=1}^{\infty} |a_n| \; ||g_n|| \; ||v^*(t)||^n \qquad (4.45)$$

where in (4.44) and (4.45). we have:

$$||v^*(t)|| = \sup_{0\le k<\infty} |v(k)| \qquad (4.46)$$

A similar expression holds for $||x^*(t)||$.

Definition (4.46) is based on the assumption that

$$v^*(t) = 0 \qquad \text{for} \quad t < 0 \tag{4.47}$$

Now, if the nonlinear discrete-data system (4.25) is BIBO stable*, then the R.H.S. of (4.45) is convergent. If also the system possesses a unique solution*, then (4.45) shows that the system is analytic and equation (4.39) is a Volterra series[20,37].

4.4 Nonlinear Discrete-Data Systems with ZOH

The system to be considered in this section has the same configuration of Fig. 4.1. However, in the present case the linear plant G_o contains a zero order hold device. Thus, the forward path, being linear, can be represented by a difference equation of the form (3.43):

$$a_n x(k+n) + a_{n-1}x(k+n-1) + \ldots + a_o x(k)$$
$$= c_m e(k+m) + c_{m-1}e(k+m-1) + \ldots + c_o e(k) \tag{4.48}$$

where $m < n$ for a physical plant; and $x(k)$ and $e(k)$ are the output and error signals at $t=kT$ respectively.

The error signal $e(k)$ is given by:

$$e(k) = y(k) - \phi(x(k)) \tag{4.49}$$

Substituting (4.49) into (4.48) and making use of (4.2) we find:

$$a_n x(k+n) + a_{n-1}x(k+n-1) + \ldots + a_{m+1}x(k+m+1) + (a_m+\alpha_m)x(k+m)$$
$$+ (a_{m-1}+\alpha_{m-1})x(k+m-1) + \ldots + (a_o+\alpha_o)x(k)$$
$$= c_m e_1(k+m) + c_{m-1}e_1(k+m-1) + \ldots \; c_o e_1(k) \tag{4.50}$$

where $\alpha_i = \gamma_1 c_i$, $i = 0, 1, 2, \ldots, m$ (4.51)

$$e_1(j) = y(j) - \phi_1(x(j)) \tag{4.52}$$

---

* See section 4.5.

Equation (4.50) describes the nonlinear discrete-data system under investigation.

It can be readily seen that equation (4.50) has the form (3.43), thus its solution is:

$$x^*(t) = \sum_{k=0}^{\infty} \{<\underline{b}(k),\underline{x}(0)> - <\underline{f}(k),\underline{e}_1(0)> + \sum_{j=0}^{k} g(k-j)e_1(j)\}\delta(t-kT) \tag{4.53}$$

Note here the $e_1(0)$ is a known vector, the i^{th} element of which is given by

$$e_{1i}(0) = y_i(0) - \phi_1(x_i(0)), \quad i=1, 2, \ldots , m \tag{4.54}$$

so that

$$\underline{e}_1(0) = \underline{y}(0) - \underline{\phi}_1(x'(0)) \tag{4.55}$$

In equations (4.53) to (4.55), $\underline{x}(0)$, $\underline{y}(0)$ and $\underline{b}(k)$ are defined by equations (3.45), (3.46), (3.19) respectively. $\underline{f}(k)$ is defined by relations similar to (3.17) and (3.19). $\underline{\phi}_1(x'(0))$ is defined by:

$$\underline{\phi}_1(x'(0)) = [\phi_1(x_1(0)) \quad \phi_1(x_2(0)) \quad . . . \quad \phi_1(x_m(0))]^T \tag{4.56}$$

and

$$\underline{x}'(0) = [x_1(0) \quad x_2(0) \quad . . . \quad x_m(0)]^T \tag{4.57}$$

The discrete impulse response is obtained from:

$$g(j) = Z^{-1}(G(z)) \tag{4.58}$$

where

$$G(z) = \frac{c_m z^m + c_{m-1}z^{m-1} + \ldots + c_1 z + c_o}{a_n z^n + a_{n-1}z^{n-1} + \ldots + (a_m+\alpha_m)z^m + \ldots + (a_o+\alpha_o)} \tag{4.59}$$

Substituting equations (4.52) and (4.55) into (4.53) we get:

$$x^*(t) = \sum_{k=0}^{\infty} \{<\underline{b}(k),\underline{x}(0)> - <\underline{f}(k),\underline{y}(0)> - <\underline{f}(k),\ \underline{\phi}_1(x'(0))>$$

$$+ \sum_{j=0}^{k} g(k-j)y(j) - \sum_{j=0}^{k} g(k-j)\phi_1(x(j))\}\delta(t-kT) \quad (4.60)$$

It can be seen that if we neglect the nonlinear element $\phi_1(x)$ from (4.60), we obtain the solution, $v^*(t)$, of the linearized system given by (3.49), so that we get:

$$x^*(t) = v^*(t) - \sum_{k=0}^{\infty} \{<\underline{f}(k),\underline{\phi}_1(x'(0))> + \sum_{j=0}^{k} g(k-j)\phi_1(x(j))\}\delta(t-kT) \quad (4.61)$$

or, using functional representation

$$x^*(t) = v^*(t) - w^*(t) - \beta^*\phi_1^*(x(k)) \quad (4.62)$$

where

$$w^*(t) = \sum_{k=0}^{\infty} <\underline{f}(k),\underline{\phi}_1(x'(0))>\ \delta(t-kT) \quad (4.63)$$

and β^* is the convolution summation operator defined by equation (3.27), viz.:

$$\beta^* u^*(t) = \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(k-j)u(j)] \quad (t-kT) \quad (4.64a)$$

$$= \sum_{k=0}^{\infty} [\sum_{j=0}^{k} g(j)u(k-j)]\ \delta(t-kT) \quad (4.64b)$$

If we now substitute for $\phi_1(x(k))$ into (4.62) one gets:

$$x^*(t) = v^*(t) - w^*(t) - \beta^*[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)] \quad (4.65)$$

With the input $y^*(t)$ fixed equation (4.65) defines the operator equation :

$$x^*(t) = U x^*(t)$$

where U is a nonlinear operator given by:

$$\dot{U}\,x^* = v^*(t) - w^*(t) - \beta^*\left[\sum_{i=2}^{M} \gamma_i x^{i^*}(t)\right] \tag{4.66}$$

Now if we let $\gamma_3 = \varepsilon$, $\varepsilon > 0$, and $\gamma_i = 0$, $i = 2, 4, 5, \ldots, M$, equations (4.65) and (4.66) become:

$$x^*(t) = v^*(t) - w^*(t) - \varepsilon\beta^* x^{3^*}(t) \tag{4.67}$$

and

$$\dot{U}\,x^* = v^*(t) - w^*(t) - \varepsilon\beta^* x^{3^*}(t) \tag{4.68}$$

Equation (4.67) can be solved by iteration, starting with $q^*(t)$ as a first approximation, in the manner described in section 4.3 to get the Volterra series:

$$\begin{aligned} x(k) = q(k) &- \varepsilon \sum_{j=0}^{k} g(k-j)\, q^3(j) \\ &+ 3\varepsilon^2 \sum_{j=0}^{k} \sum_{j_1=0}^{k} g(k-j) g(j-j_1)\, q^2(j)\, q^3(j_1) \\ &+ \ldots + \ldots \end{aligned} \tag{4.69}$$

where $q(k) = v(k) - w(k)$ (4.70)

Note $q(k)$ is a known sequence since both $v(k)$ and $w(k)$ are known. Note also that (4.69) is similar to (4.39) with $q(k)$ taking the place of $v(k)$, and thus it can be justified in the same way.

4.5 Convergence of the Discrete Volterra Series and Stability of Nonlinear Discrete-Data Systems.

In this section we will investigate the stability of the nonlinear discrete-data systems considered in this chapter. This will be done by considering the convergence of the Volterra series representing system response. Convergence will be established by using the contraction mapping theorem. The theorem is stated next. The proof of the theorem is given in reference 25.

THEOREM Let F be a Banach space, and let A be a mapping of F into itself. Let S be a sphere contained in F of centre u_o, and radius d, such that:

$$\text{(i)} \quad ||Au_1 - Au_2|| \leq \lambda \, ||u_1 - u_2|| \tag{4.71}$$

for every u_1, $u_2 \in S$ and some λ: $0<\lambda<1$

$$\text{(ii)} \quad ||Au_o - u_o|| < (1 - \lambda)d \tag{4.72}$$

then the mapping A possesses a unique fixed point u in S, that is a point for which

$$Au = u \tag{4.73}$$

Condition (i) will be called the contraction condition, and condition (ii) the fixed point condition.

Now, to illustrate the application of this theorem to the investigation of system stability, it is most convenient to consider a specific example.

Illustrative Example

Consider the nonlinear discrete-data system shown in Fig. 4.2.

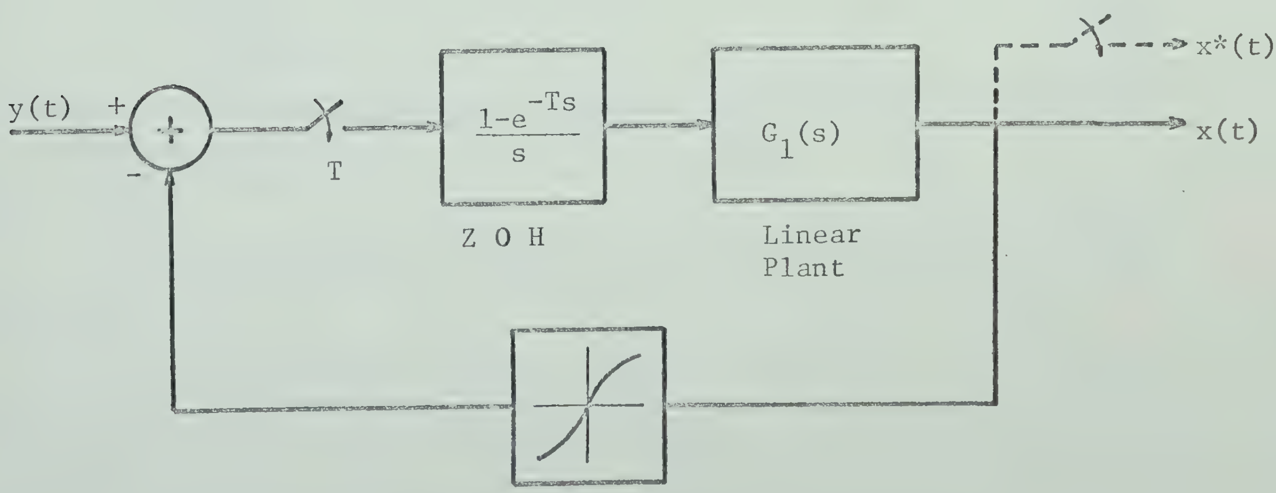

Fig. 4.2 Nonlinear Discrete-Data System
For Illustrative Example

The system contains a nonlinear element located in the feedback path which is described by the gain function:

$$\phi(x) = x + \varepsilon x^3 \tag{4.74}$$

The forward path of the system consists of a sampler with period T=1 second, followed by a zero order hold and a first-order linear plant whose transfer function, $G_1(s)$, is given by:

$$G_1(s) = \frac{1}{s+1} \tag{4.75}$$

The z-transfer function of the forward path is, thus, given by:

$$G_o(z) = Z[\frac{1-e^{-Ts}}{s} \cdot \frac{1}{s+1}]$$

$$= \frac{1-e^{-T}}{z-e^{-T}}$$

$$= \frac{0.632}{z-0.368} \tag{4.76}$$

The linearized system is obtained by letting $\varepsilon=0$ in (4.74). The overall z-transfer function of the linearized system is given by:

$$G(z) = \frac{G_o(z)}{1+G_o(z)}$$

$$\doteq \frac{0.632}{z+0.264} \tag{4.77}$$

clearly, the linearized system is stable since its only pole lies inside the unit circle $|z| = 1$ in the z-plane[38].

The difference equation representing the nonlinear system is readily obtained as:

$$x(k+1) + 0.264\, x(k) + 0.632\, \varepsilon\, x^3(k) = 0.632\, y(k) \tag{4.78}$$

where y(k) is the value of the input function, y(t), at time t=kT=k. We note here that the difference equation representing the nonlinear system, which contains a ZOH, is of the form (4.25) rather than (4.50).

This is because the linear plant $G_1(s)$ is a first order plant. The solution of the system is, thus:

$$x(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)\, x^3(j) \tag{4.79}$$

where v(k) is the solution of the linearized system. It is given by:

$$v(k) = \langle b(k), x(0)\rangle + \sum_{j=0}^{k} g(k-j) y(j) \tag{4.80}$$

where g(k), the impulse response of the linearized system, is given by:

$$g(k) = \sum \text{Residues of } G(z)\, z^{k-1} \text{ at the poles of } G(z)\, z^{k-1} \tag{4.81}$$

$$= \begin{cases} 0 & \text{for } k = 0 \\ 0.632(-0.264)^{k-1} & \text{for } k > 0 \end{cases} \tag{4.82}$$

and

$$b(k) = Z^{-1}\left[\frac{z}{Z + 0.264} \right]$$

$$= (-0.246)^k \quad , \quad k \geq 0 \tag{4.83}$$

Equation (4.80) can be written as:

$$v^*(t) = f^*(t) + \beta^* y^*(t) \tag{4.84}$$

where

$$f^*(t) = b^*(t)\, x(0) \tag{4.85}$$

Taking the norm on both sides of (4.84), we get

$$V^* = ||v^*(t)|| = \sup_{0 \leq k < \infty} |v(k)|$$

$$\leq F^* + H^* Y^* \tag{4.86}$$

where

$$F^* = ||f^*(t)|| = \sup_{0 \leq k < \infty} |f(k)| = |x(0)| \tag{4.87}$$

$$Y^* = ||y^*(t)|| \tag{4.88}$$

$$H^* = ||\beta^*|| = \sum_{k=0}^{\infty} |g(k)|$$

$$= \sum_{k=1}^{\infty} |0.632(-0.264)^{k-1}|$$

$$= 0.632 \sum_{k=1}^{\infty} (0.264)^{k-1}$$

$$= 0.86 \tag{4.89}$$

Now, equation (4.79) can be written as:

$$x^*(t) = v^*(t) - \varepsilon\beta^* x^{3^*}(t) \tag{4.90}$$

$$= U\, x^*(t) \tag{4.91}$$

To apply the contraction condition (4.71), we let $x_1^*(t)$ and $x_2^*(t)$ be two elements of the sphere $S \subset F$, centred at $v^*(t)$ and of radius 'd'. The choice of $v^*(t)$ as the centre of S is justified on the basis that the nonlinear system may be looked upon as a perturbed version of the linearized system, so that if it has a fixed point for a given $y^*(t) \in F$ and some initial conditions, then this fixed point is likely to be in the neighbourhood 'd' of the solution of the linearized system for the same input $y^*(t) \in F$ and same initial conditions.

Now the contraction condition is:

$$||\, U x_1^*(t) - U x_2^*(t)|| \leq \lambda\, ||x_1^*(t) - x_2^*(t)|| \;,\; 0<\lambda<1. \tag{4.92}$$

From equation (4.90):

$$||\, U x_1^*(t) - U x_2^*(t)||$$
$$= ||\varepsilon\beta^* x_2^{3^*}(t) - \varepsilon\beta^* x_1^{3^*}(t)||$$
$$\leq |\varepsilon|\, ||\beta^*||\; ||x_2^3(t) - x_1^{3^*}(t)||$$
$$\leq |\varepsilon|\; H^*\; ||x_2^*(t) - x_1^*(t)|| \sum_{j=0}^{2} ||x_2^*(t)||^{2-j}||x_1^*(t)||^j \tag{4.93}$$

Since $x_1^*(t)$, $x_2^*(t)$ εS:

$$||x_1^*(t)||, ||x_2^*(t)|| \leq ||v^*(t)|| + d$$

$$= V^* + d \qquad (4.94)$$

then:

$$|| U x_1^*(t) - U x_2^*(t)|| \leq 3 |\varepsilon| H^*(V^* + d)^2 ||x_2^*(t) - x_1^*(t)|| \qquad (4.95)$$

since $||x_2^*(t) - x_1^*(t)|| = ||x_1^*(t) - x_2^*(t)||$, (4.95) is equivalent to

$$3 |\varepsilon| H^*(V^* + d)^2 \leq \lambda \qquad (4.96)$$

We now consider the fixed point condition (4.72),

$$|| U v^*(t) - v^*(t)|| \leq (1 - \lambda)d \qquad (4.97)$$

From equations (4.90) and (4.91), it follows:

$$U v^*(t) = v^*(t) - \varepsilon\beta^* v^{*3}(t) \qquad (4.98)$$

Transposing $v^*(t)$ to the L.H.S. and taking the norms on both sides, one gets:

$$\begin{aligned} || U v^*(t) - v^*(t)|| &= ||-\varepsilon\beta^* v^{*3}(t)|| \\ &\leq |\varepsilon| \; ||\beta^*|| \; ||v^*(t)||^3 \\ &= |\varepsilon| \; H^* \; V^{*3} \end{aligned} \qquad (4.99)$$

Hence (4.97) is satisfied if:

$$|\varepsilon| \; H^* \; V^{*3} \leq (1 - \lambda)d \qquad (4.100)$$

Rao[23] considered the same class of systems, but with zero initial conditions. He obtained the same inequalities (4.96) and (4.100) with H^*Y^* taking the place of V^*. For the particular example considered here, he obtained as a bound for the input, such that the nonlinear system is BIBO stable, the following:

$$Y^* < 0.435 \; |\varepsilon|^{-\frac{1}{2}} \qquad (4.101)$$

which corresponds to:

$$H^*Y^* < 0.374 \; |\varepsilon|^{-\frac{1}{2}} \qquad (4.102)$$

This means that, for the nonlinear system to be BIBO stable, the response of the linearized system must be bounded by the quantity in the R.H.S. of (4.102). In our case, this corresponds to:

$$V^* \leq F^* + H^* Y^* < 0.374 \; |\varepsilon|^{-\frac{1}{2}} \tag{4.103}$$

which gives as a bound for the input:

$$\begin{aligned} Y^* &< \frac{1}{H^*} [0.374|\varepsilon|^{-\frac{1}{2}} - F^*] \\ &= \frac{1}{H^*} [0.374 \; |\varepsilon|^{-\frac{1}{2}} - |x(0)|] \end{aligned} \tag{4.104}$$

If the system is undriven, $Y^* = 0$ and we get:

$$F^* = |x(0)| < 0.374 \; |\varepsilon|^{-\frac{1}{2}} \tag{4.105}$$

which is the bound on $x(0)$ such that the nonlinear system is asymptotically stable.

For $\varepsilon = 0.1$, conditions (4.104) and (4.105) become, respectively:

$$Y^* < 1.163 \; [1.183 - |x(0)|] \tag{4.106}$$

$$|x(0)| < 1.183 \tag{4.107}$$

Clearly, if we let $x(0) = 0$ in (4.104) we get

$$Y^* < 1.375 \tag{4.108}$$

which is the result obtained by Rao[23].

Now we recall that the response of the nonlinear system (4.78) is found to be:

$$\begin{aligned} x(k) = v(k) &- \varepsilon \sum_{j=0}^{k} g(k-j) \; v^3(k) \\ &+ 3\varepsilon^2 \sum_{j=0}^{k} \sum_{j_1=0}^{k} g(k-j)g(j-j_1)v^2(j)v^3(j_1) \\ &- 3\,\varepsilon^3 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} g(k-j)g(j-j_1)g(j-j_2)v(j)v^3(j_1)v^3(j_2) \end{aligned}$$

$$+ \varepsilon^4 \sum_{j=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \sum_{j_3=0}^{k} [g(k-j)g(j-j_1)g(j-j_2)g(j-j_3)v^3(j_1)v^3(j_2) v^3(j_3)]$$

$$+ \ldots + \ldots \tag{4.109}$$

By taking the norms on both sides we found:

$$X^* \leq V^* + \sum_{n=1}^{\infty} |a_n| \, ||g_n|| \, V^{*n} \tag{4.110}$$

Hence, since (4.96) and (4.100) are satisfied, the iterative series solution found from (4.79) is a unique convergent series. Furthermore, since the solution can be put into the form (4.110), that series is the Volterra series solution for the system considered[24].

CHAPTER V

RESPONSE OF NONLINEAR DISCRETE-DATA SYSTEMS BETWEEN SAMPLING INSTANTS

5.1 Introduction

Since the response of sampled-data systems is continuous in time, it is very important to know how they behave between sampling instants. A system may seem to be stable judged by its response at sampling instants, though it may exhibit an unstable response between sampling instants.

In this chapter nonlinear discrete-data systems will be considered. Relations that determine the intersampling response of the two types of systems considered in chapter IV will be derived. It will be shown that these responses are obtained in terms of an infinite series. Stability of the systems between sampling instants is considered at the same time.

5.2 Deriviation of Response Equations

The system considered is described in section 4.2 and is shown in Fig. 5.1. The linear plant G_o will be defined by its transfer function $G_o(s)$. It may or may not include a ZOH device. The nonlinear element N in the feedback path is described by relations (4.1) and (4.2), viz.:

$$\phi(x) = \gamma_1 x + \sum_{i=2}^{M} \gamma_i x^i \qquad (5.1)$$

$$= \gamma_1 x + \phi_1(x) \qquad (5.2)$$

It is well known[39] that the intersampling response of linear discrete-data systems can be determined from the relation:

$$x(k+m) = \sum_{j=0}^{k} g(k+m-j)\ y(j) \tag{5.3}$$

where $x(k+m)$ is the response of the system at $t = (k+m)T$, $0 \leq m \leq 1$, T is the sampling period; $y(j)$ is the input at $t=jT$ and $g(k+m)$ is given by:

$$g(k+m) = L^{-1}[G(s)]\Big|_{t=(k+m)T} \tag{5.4}$$

where $L^{-1}[\quad]$ represents the operation of taking the inverse Laplace-transform.

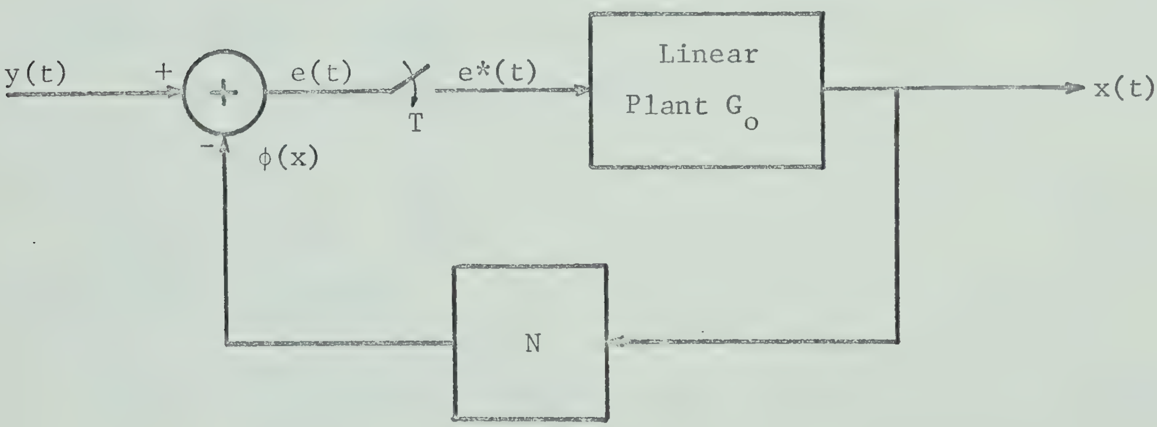

Fig. 5.1 Nonlinear Discrete-Data System For The Purpose of Determination of Intersampling Response.

Applying relation (5.3) to the forward path of the system of Fig. 5.1 we find

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)\ e(j) \tag{5.5}$$

where $e(j)$ is the error signal at $t=jT$. It is given by:

$$e(j) = y(j) - \phi(x(j)) \tag{5.6}$$

Substituting (5.6) into (5.5), and making use of (5.2), equation (5.5) becomes:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - \gamma_1 x(j) - \phi_1(x(j))] \tag{5.7}$$

Note that if $\phi_1(x(j))$ is zero, the system will reduce to the linearized system, whose response will be

$$v(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - \gamma_1 v(j)] \tag{5.8}$$

The intersampling response of the nonlinear discrete-data system is now readily obtained by substituting the response of the system at sampling instants in the R.H.S. of (5.7)

For the particular nonlinearity considered in chapter IV which is:

$$\phi(x(k)) = x(k) + \varepsilon x^3(k) \tag{5.9}$$

equation (5.7) becomes:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - x(j) - \varepsilon x^3(j)] \tag{5.10}$$

Now, for a system that does not contain a ZOH device, the response at sampling instants is given by equation (4.39), viz.:

$$x(k) = v(k) - \varepsilon \sum_{j=0}^{k} g(k-j)v^3(j) + \ldots \tag{5.11}$$

Substituting (5.11) into (5.10) one gets:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - v(j) + \varepsilon \sum_{j_1=0}^{j} g(j-j_1)v^3(j_1) - \ldots$$
$$- \varepsilon\, (v(j) - \varepsilon \sum_{j_1=0}^{j} g(j-j_1)v^3(j_1) - \ldots)^3] \tag{5.12}$$

Expanding equation (5.12) we get:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)\ [y(j) - v(j)]$$
$$- \varepsilon \sum_{j=0}^{k} g_o(k+m-j)[v^3(j) - \sum_{j_1=0}^{j} g(j-j_1)v^3(j_1)]$$
$$+ \varepsilon^2 \sum_{j=0}^{k} g_o(k+m-j)[\sum_{j_1=0}^{j} g(j-j_1)v^2(j)v^3(j_1) - 3 \sum_{j_1=0}^{j} \sum_{j_2=0}^{j} g(j-j_1)g(j-j_2)\, v^2(j_1)v^3(j_2)]$$
$$+ \ldots + \ldots \tag{5.13}$$

$$= v(k+m) - \varepsilon \sum_{j=0}^{k} g_o(k+m-j)[v^3(j) - \sum_{j_1=0}^{j} g(j-j_1)v^3(j_1)]$$
$$+ \varepsilon^2 \sum_{j=0}^{k} g_o(k+m-j)[\sum_{j_1=0}^{j} g(j-j_1)v^2(j)v^3(j_1) - 3 \sum_{j_1=0}^{j} \sum_{j_2=0}^{j} g(j-j_1)g(j-j_2)\, v^2(j_1)v^3(j_2)]$$
$$+ \ldots + \ldots \tag{5.14}$$

where $v(k+m)$ is the response of the linearized system between sampling instants. For this particular case it is given by:

$$v(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - v(j)] \tag{5.15}$$

Equation (5.14) gives the response of the nonlinear discrete-data system between sampling instants when the system does not contain a ZOH device.

The series corresponding to (5.14) for the case of systems with ZOH is obtained by replacing v(k) in (5.14) by q(k) from equation (4.70).

5.3 Stability Considerations

Let us consider first the stability of the linearized system. The response of this system between sampling instants is given by:

$$v(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - \gamma_1 v(j)] \tag{5.16}$$

It will be assumed that the linearized system is stable at sampling instants, then stability of (5.16) is assured if:

$$\sum_{k=0}^{\infty} |g_o(k+m)| < \infty \tag{5.17}$$

which implies that

$$\lim_{k\to\infty} |g_o(k+m)| = 0 \tag{5.18}$$

Note that, by stability we imply BIBO stability as well as asymptotic stability.

Now, we proceed to the case of nonlinear systems. Stability of nonlinear discrete-data systems follows directly from the stability of these systems at sampling instants, and the stability of the linearized systems between sampling instants. This can be shown as follows:

The intersampling response of nonlinear discrete-data systems is given by equation (5.7), viz.:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - \gamma_1 x(j) - \phi_1(x(j))] \tag{5.19}$$

substituting for $\phi_1(x)$ one gets:

$$x(k+m) = \sum_{j=0}^{k} g_o(k+m-j)[y(j) - \gamma_1 x(j) - \sum_{i=2}^{M} \gamma_i x^i(j)] \tag{5.20}$$

Let us now test for BIBO stability. According to our assumptions,

$$\sup_{0 \le k < \infty} |x(k)| < \infty \tag{5.21}$$

$$\sup_{0 \le k < \infty} |y(k)| < \infty \tag{5.22}$$

and inequality (5.17) is satisfied. By taking the absolute value of both sides of (5.20) we get:

$$|x(k+m)| < \{ \sup_{0 \le k < \infty} |y(k)| + |\gamma_1| \sup_{0 \le k < \infty} |x(k)| + \sum_{i=2}^{M} |\gamma_i| [\sup_{0 \le k < \infty} |x(k)|]^i \} \sum_{k=0}^{\infty} |g_o(k+m)| \tag{5.23}$$

clearly, the R.H.S. of (5.23) is bounded, and since (5.23) is true for all k and m, $0 \le m \le 1$, it follows that

$$\sup_{\substack{0 \le k < \infty \\ 0 \le m \le 1}} |x(k+m)| < \infty \tag{5.24}$$

which means that the intersampling response of the nonlinear system is bounded. It also means that the series (5.14) is convergent.

Similar argument can be used for testing asymptotic stability.

5.4 Illustrative Example

In this example we will consider the same system considered in the example of the previous chapter. The system is shown in Fig. 4.2. The transfer function of the forward path is given by:

$$G_o(s) = \frac{1-e^{-Ts}}{s(s+1)} \tag{5.25}$$

The intersampling impulse response of the forward path will be obtained in the following way:

The modified z-transform of $G_o(s)$ is found to be:

$$G_o(z,m) = \frac{z(1-e^{-mT})-e^{-T}+e^{-mT}}{z(z - e^{-T})} \tag{5.26}$$

Taking the inverse modified z-transform of equation (4.28), we get:

$$\begin{aligned} g_o(n-1+m) &= Z_m^{-1}[G_o(z,m)] \\ &= \sum \text{Residues of } G_o(z,m)z^{n-1} \text{ at the poles} \\ &\qquad \text{of } G_o(z,m)z^{n-1} \end{aligned} \tag{5.27}$$

(See reference 38).

For n=1, $G_o(z,m)$ has two poles at $z=0$ and $z=e^{-T}$, so that:

$$g_o(0+m) = \frac{e^{-T}-e^{-mT}}{e^{-T}} + (1-e^{-mT}) - \frac{e^{-T}-e^{-mT}}{e^{-T}}$$

$$= (1-e^{-mT}) \tag{5.28}$$

For n>1, $G_o(z,m)$ has only one pole at $z=e^{-T}$, so that:

$$g_o(n-1+m) = e^{-mT}\ (1-e^{-T})e^{-T(n-2)} \tag{5.29}$$

Letting k=n-1, we get:

$$g_o(k+m) = e^{-mT}(1-e^{-T})e^{-T(k-1)}\ , \quad k=1,2\ \ldots \tag{5.30}$$

Now, for T=1 second, we get:

$$g_o(k+m) = \begin{cases} 1 - e^{-m} & \text{for } k=0 \\ 0.632(0.368)^{k-1}e^{-m} & \text{for } k>0 \end{cases} \tag{5.31}$$

Now,

$$\begin{aligned} \sum_{k=0}^{\infty} |g_o(k+m)| &= |1-e^{-m}| + 0.632\ e^{-m} \sum_{k=1}^{\infty} (0.368)^{k-1} \\ &= |1-e^{-m}| + 0.632\ e^{-m}\frac{1}{1-0.368} \\ &= |1 - e^{-m}| + e^{-m} \end{aligned} \tag{5.32}$$

It was shown by equation (4.77) that the linearized system is stable at sampling instants. Also equation (5.32) shows that inequality (5.17) is satisfied. Hence, the linearized system is stable between sampling instants. Furthermore, the nonlinear system was shown to be stable at sampling instants in the region in which inequalities (4.96) and (4.100) are satisfied. Hence, it follows that the nonlinear system is also stable between sampling instants.

It has been found in chapter IV that the bound on the input $y^*(t)$, in order that the nonlinear system of Fig. 4.2 is BIBO stable at sampling instants, is given by inequality (4.106), viz.:

$$Y^* = ||y^*(t)|| = \sup_{0 \leq k < \infty} |y(k)| < 1.163\ (1.183 - |x(0)|) \tag{5.33}$$

The corresponding bound on the output of the nonlinear system at sampling instants is found to be[23]:

$$X^* = ||x^*(t)|| = \sup_{0 \leq k < \infty} |x(k)| < 2.5 \tag{5.34}$$

A bound on the intersampling response of the driven nonlinear system is now readily obtainable from inequality (5.23). Substituting (5.32) to (5.34) into (5.23) this bound is found to be:

$$X_m^* = \sup_{\substack{0 \leq k < \infty \\ 0 \leq m \leq 1}} |x(k+m)| < [5.435 - 1.163|x(0)|][|1-e^{-m}|+e^{-m}] \tag{5.35}$$

If the system is undriven, then $Y^*=0$ and, provided that (4.107) is satisfied, the nonlinear discrete-data system is asymptotically stable between sampling instants. The bound on its intersampling response in this case is given by:

$$X_m^* < 4.06\ [|1-e^{-m}|+e^{-m}] \tag{5.36}$$

CHAPTER VI

CONCLUSION

In this thesis we have dealt with the response of nonlinear discrete-data systems when the initial conditions are not zero. We have also considered the stability problem under these circumstances. After reviewing the main results in the field of nonlinear discrete-data system stability, non-zero initial conditions are included in a general expression for linear discrete-data system response for both systems considered. Bounded-input bounded-output stability as well as asymptotic stability of linear systems have also been discussed. The results are extended to the case of nonlinear discrete-data systems, and by solving the system response equation by iteration, a discrete Volterra type series is obtained in which non-zero initial conditions are accounted for. Uniqueness and convergence of the series solution obtained in this manner, as well as bounded-input bounded-output stability of the nonlinear system, are investigated by the use of the contraction mapping theorem (Banach fixed point theorem) of functional analysis. By equating the input of the system to zero, asymptotic stability of the nonlinear system is established in the region where the conditions of the contraction mapping theorem are fulfilled. An example is given to illustrate the results obtained.

Finally the response of nonlinear discrete-data systems between sampling instants is considered. The response takes the form of an infinite series, which is shown to be bounded. Bounded-input

bounded-output stability as well as asymptotic stability of nonlinear discrete-data systems between sampling instants are also investigated. It has been shown that the region of system stability between sampling instants is identical to the region of system stability at sampling instants, i.e. the bounds on the input and initial conditions are identical in both cases.

It is worth noting here that since the contraction condition is a strong requirement, the region obtained by this method are bound to be on the conservative side. However, unlike the method used by Alper, this method has the advantage that the series obtained is a convergent series.

In conclusion the author wishes to state that the main contribution of this thesis is the inclusion of non-zero initial conditions in the Volterra series representation of the response of a class of nonlinear discrete-data systems at sampling instants, and the determination of system response between sampling instants.

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APPENDIX

Z-TRANSFORM THEORY

The theory of the Z-transform is well known. Its development from the Laplace transform and its main theorems are discussed in the textbooks on the subject of sampled-data systems[38,40]. However, the defining equation of the Z-transform will be derived here, then its main theorems and the different methods for the determination of its inverse are stated.

A.1 The Z-Transform

A uniformly sampled signal $x^*(t)$ of uniform sampling period T, is a signal such that

$$x^*(t) = x(t) \sum_{n=0}^{\infty} \delta(t-nT)$$

$$= \sum_{n=0}^{\infty} x(nT)\ \delta(t-nT) \qquad \text{(A.1)}$$

where $x(t)$ is the continuous signal before sampling. This means that a sampled signal is a train of impulses separated from each other by T seconds and each is of height equal to the value of the sampled signal at the sampling instants. It is assumed that the continuous signal $x(t)$ is zero for $t<0$.

If now equation (A.1) is Laplace transformed, one gets:

$$x^*(s) = \sum_{n=0}^{\infty} x(nT)\ e^{-nTs} \qquad \text{(A.2)}$$

Letting $e^{Ts}=z$, equation (A.2) may be written as:

$$x^*(s)\Big|_{s=\frac{1}{T}\ln z} = x(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n} \tag{A.3}$$

Equation (A.3) defines the Z-transform of the signal x(t).

A.2 Theorems of the Z-Transform

In this section only the theorems used in the development presented will be stated.

A.2.1 Linearity

If $g_1(t)$ and $g_2(t)$ are Z-transformable functions and if a and b are two constants, then

$$Z[ag_1(t) \pm bg_2(t)] = a\, Z[g_1(t)] \pm b\, Z[g_2(t)] \tag{A.4}$$

where Z[ ] represents the process of Z-transformation.

A.2.2 Real translation

If g(t) is Z-transformable function and if n is a positive integer, T is the sampling period, then

$$Z[g(t - nT)] = z^{-n}\, Z[g(t)] \tag{A.5}$$

and

$$Z[g(t+nT)] = z^{n}\, [Z[g(t)] - \sum_{k=0}^{n-1} g(kT) z^{-k}] \tag{A.6}$$

A.3 The Inverse Z-Transform

The determination of the sampled signal from its Z-transform is as important as the determination of the Z-transform of a sampled signal. This process of passing from the Z-domain into the time domain is called "the inverse Z-transformation".

To evaluate the inverse Z-transform, one of the following three methods can be used.

A.3.1 The Inversion Formula[38*]

The sampled signal $g^*(t)$ defined by

$$g^*(t) = \sum_{n=0}^{\infty} g(nT)\, \delta(t-nT) \tag{A.7}$$

T being the sampling period, can be evaluated from its Z-transform G(z) by the use of the inversion integral

$$g(nT) = \frac{1}{2\pi j} \oint_{\Gamma} G(z)\, z^{n-1}\, dz \tag{A.8}$$

where Γ is a circle of radius $z=e^{cT}$ centered at the origin of the z-plane, and c is a constant such that all the poles of G(z) are enclosed by the circle.

By the use of Cauchy's residues theorem equation (A.8) can be rewritten as:

$$g(nT) = \text{Sum of the residues of } G(z)\, z^{n-1} \text{ at the poles of } G(z)\, z^{n-1} \tag{A.9}$$

A.3.2 Power Series Expansion

Here the Z-transform G(z) is written in the form:

$$G(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_o}{c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_o} \tag{A.10}$$

where for physical systems $m \leq n$.

By performing long division G(z) may be written in the form:

$$G(z) = a_o + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n} + \ldots \tag{A.11}$$

Now by taking the z-transform of equation (A.7) term by term one gets:

$$G(z) = g(o) + g(T) z^{-1} + g(2T) z^{-2} + \ldots + g(nT) z^{-n} + \ldots \tag{A.12}$$

---

* see in particular pp. 66-68. See also Ref. 40, pp. 176-179.

By comparing equations (A.11) and (A.12) we can conclude that when $G(z)$ is expanded into a power series of z^{-1}, the coefficient of z^{-n} in the expansion corresponds to the value of the signal $g(t)$ at $t=nT$.

A.3.3 Partial Fraction Expansion

Here $\frac{G(z)}{z}$ is expanded into partial Fractions of the form $\frac{a_i}{z-c_i}$ where the a_i's are constants and the c_i's are the poles of $\frac{G(z)}{z}$. Then $G(z)$ will be formed of terms of the form $\frac{a_i z}{z-c_i}$. By the use of z-transform tables the time function corresponding to each term is obtained to get a time function $g(t)$ corresponding to $G(z)$. This time function, however, is of significance only at sampling instants.

It should be noted that any of the above three methods evaluates the sampled signal $g^*(t)$. They do not give any information about the behaviour of the signal $g(t)$ between sampling instants.